

Ergodic properties of random billiards driven by thermostats.

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Abstract: We consider a class of mechanical particle systems interacting with thermostats. Particles move freely between collisions with disk-shaped thermostats arranged periodically on the torus. Upon collision, an energy exchange occurs, in which a particle exchanges its tangential component of the velocity for a randomly drawn one from the Gaussian distribution with the variance proportional to the temperature of the thermostat. In the case when all temperatures are equal one can write an explicit formula for the stationary distribution. We consider the general case and show that there exists a unique absolutely continuous stationary distribution. Moreover under rather mild conditions on the initial distribution the corresponding Markov dynamics converges to the equilibrium with exponential rate. One of the main technical difficulties is related to a possible overheating of moving particle. However as we show in the paper non-compactness of the particle velocity can be effectively controlled.

1. Introduction

Rigorous derivations of macroscopic heat conduction laws from microscopic dynamics of mechanical models require good mixing properties and fast convergence of initial distributions to the invariant measure(s). For many such systems in non-equilibrium, e.g. when a system is coupled to two or more unequal heat reservoirs, pure existence of invariant measures is a nontrivial and open question due to non-compactness of the phase spaces. It is relatively easy to envision scenarios under which a particle freezes or heats up, which may push initial distributions towards zero or infinite energy levels and ultimately violate existence of the invariant measures. Tightness arguments are required in order to show

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that such scenarios occur with zero probability and invariant measures indeed exist. Even stronger controls are required to obtain mixing properties.

Mechanical particle systems coupled to heat reservoirs has seen a renewed interest and activity over the last decade. Various examples were introduced in [1, 3, 4, 5, 7, 8, 9]. Rigorous results on ergodicity and absolute continuity of invariant measures assuming existence for some of these systems have been obtained [1, 5, 4, 11]. These results, however, rely on sample paths with infinitesimally low probability of occurrence giving no control on times and rates and cannot be used to obtain mixing properties or even existence of the invariant measures. We present an example of a simple particle system interacting with thermostats for which we are able to control more regular sample paths.

The example is motivated as follows: Consider a system of N non-interacting particles at various velocities bouncing elastically off the walls of a bounded domain. We assume for visualization purposes that N is very large and the system is at temperature T_0 in the following sense: kinetic energies of the particles are distributed with the Gibbs distribution with parameter $\beta_0 = \frac{1}{T_0}$, i.e. the probability that a given particle has kinetic energy dE near E is approximately $ce^{-\beta_0 E} dE$.

Let us introduce a thermostat into the system set at a different temperature $T_1 \neq T_0$ such that when a particle collides with the thermostat, an energy exchange occurs. That is, upon collision, the thermostat absorbs part of the particle's energy, which depends on the angle of the collision, and the particle acquires an energy E from the thermostat drawn from Gibbs distribution with parameter β_1 , where $\beta_1 = \frac{1}{T_1}$. Over time, such a system is expected to settle at temperature T_1 , i.e. the initial Gibbs distribution with parameter β_0 is expected to converge to the Gibbs distribution with parameter β_1 . The questions of interest are whether the Gibbs distribution with parameter β_1 is indeed the unique invariant measure for the system to which all (or almost all) initial distributions converge, and if so, at which rate.

Now let us add another thermostat at yet a different temperature $T_2 \neq T_1$. In this case the system is *not* in thermal equilibrium. Does an invariant measure exist for such a system? Is it unique and if so, do reasonable initial distributions converge to it and what are the rates of convergence? Similar questions may be asked in the presence of more than two thermostats at different temperatures.

In the absence of particle interactions the system with many particles is simply the product of one particle systems. The dynamics is described by a continuous-time Markov Process, which is deterministic apart from collisions with thermostats and upon a collision of a particle with a thermostat, a random perturbation occurs. This degeneracy of the Markov process allows to restrict the study to the discrete time dynamics on the collision manifold.

For the resulting discrete-time Markov chain we show that, under certain geometric assumptions, there exists an invariant measure and it is unique (ergodic), absolutely continuous with respect to Lebesgue measure, and mixing with exponential rates. We also conclude that reasonable initial distributions converge to the invariant measure exponentially fast with control on the rates. It follows that for the original Markov process there exists an invariant measure and this measure is unique (ergodic) and absolutely continuous. Mixing and convergence of initial distributions to the invariant measure for the Markov process do not follow directly since under some scenarios particles may move extremely slow.

Though ergodicity guarantees that they will eventually speed up, bounds on the times must be obtained in order to show mixing. We leave the investigation of mixing properties of the Markov process for future work.

The proof of mixing properties in the discrete case uses general state Markov chain machinery, in particular, Harris' Ergodic Theorem [6,10]. The theorem requires two things: to produce a non-negative function V on the phase space which, on average, decreases geometrically under the push forwards of the dynamics, and, given such a V , to show minorization or Doeblin's condition on certain level set of V . The first condition guarantees that the dynamics enters the 'center' of the phase space, a certain level set of V , with good control on the rates; and, once at the 'center', coupling is guaranteed by the minorization condition. Those two conditions imply existence and uniqueness of the invariant measure with exponential mixing rates and exponential convergence of reasonable initial distributions to that invariant measure.

We believe that most of our results can be generalized to the case of thermostats of a general smooth convex shapes rather than disks. It is also interesting and tempting to extend the methods of this paper to the 3-dimensional case. We are planning to address these and other open questions in the near future.

We describe precise settings in section 2, then state the results and outline the proofs in section 3. In section 4 we describe the potential V , and in section 5 we establish the minorization condition.

2. Settings

Let $\Gamma = T^2 \setminus \cup_{i=1}^p D_i$ be a bounded horizon billiard table with finitely many non-intersecting circular obstacles D_1, \dots, D_p having radii R_1, \dots, R_p . An example of such configuration is shown in Fig. 1. Particles move around in Γ . Apart from collisions with the boundary $\partial\Gamma$, the particles move freely with constant velocities and do not interact with each other. Each obstacle D_i , $1 \leq i \leq p$, is assigned certain parameter β_i which represents the inverse temperature and plays a role of a thermostat.

Upon a collision of a particle with a thermostat D_i , let $v^- = (v_\perp^-, v_t^-)$ be the decomposition of the particle's velocity into the normal and the tangential components with respect to the boundary of the thermostat ∂D_i . After the collision, the normal component of the velocity changes sign, $v_\perp^+ = -v_\perp^-$, while the tangential component is absorbed by the thermostat and a new tangential component v_t^+ is drawn from the distribution $\sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t^+$. The outgoing velocity has a decomposition $v^+ = (v_\perp^+, v_t^+)$. This type of energy exchange was used in [9].

Since particles do not interact with each other, we can reduce our discussion to studying the system of only one particle. The Phase Space of such a systems is

$$\tilde{\Omega} = \{(x, v) : x \in \Gamma, v \in R^2\} / \sim$$

where \sim is an identification of points on the collision manifold that correspond to incidence and reflection with the choice of the old, incoming, velocities $v^- = (v_\perp^-, v_t^-)$.

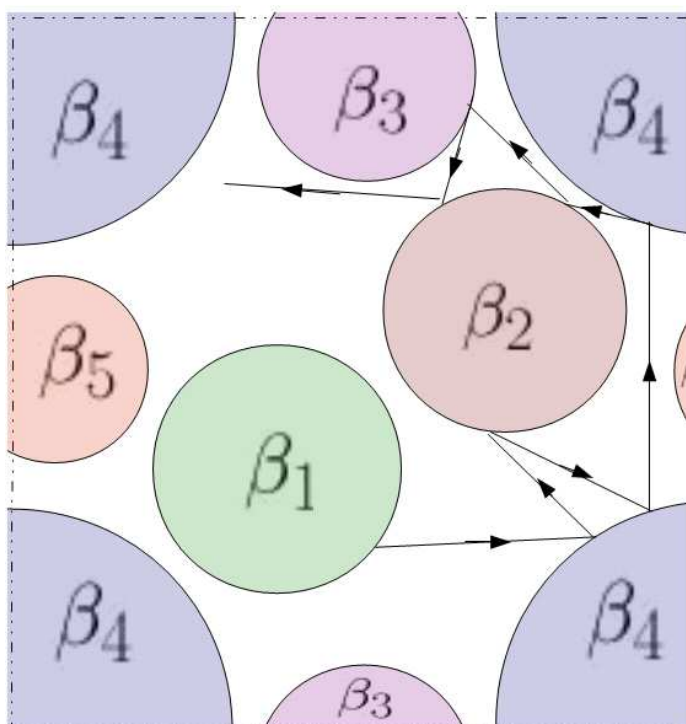


Fig. 1. Geometric Configuration

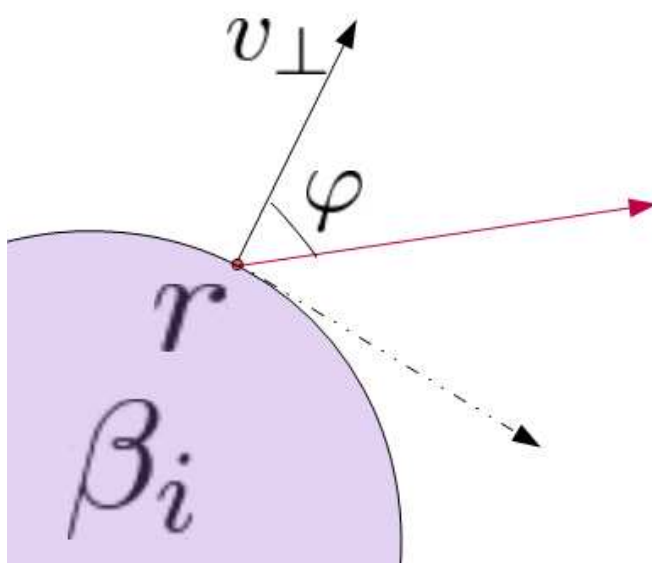


Fig. 2. Discrete Dynamics Coordinates

Then the dynamics is described by a Markov Process $\tilde{\Phi}_\tau$ on $\tilde{\Omega}$, which is deterministic between collisions with thermostats with random kicks at the moments of collisions with $\partial\Gamma$.

As usual for such systems, it is convenient to pass to a discrete dynamics on the boundary $\partial\Gamma$ of Γ . Note that at the moment of collision we chose to keep old, incoming velocities. Given such a choice, to describe the discrete dynamics it is enough to keep track of the sequence of collision points $r_n \in \partial\Gamma$ (parameterized by arc-length) and the absolute value of the normal velocity $v_\perp(n)$ at the moment of collision. Using randomly generated tangential velocity $v_t(n)$ drawn from the probability distribution $\sqrt{\frac{\beta}{\pi}} e^{-\beta v_t^2} dv_t$, one determines the outward velocity and, hence, the next point of collision r_{n+1} and the next normal velocity $v_\perp(n+1)$. This procedure defines a Markov chain Φ on $\Omega = \partial\Gamma \times [0, \infty)$.

For our future analysis it would be more convenient to use a geometric description similar to one used in billiards. Namely, instead of the random variable $v_t(n)$, we shall consider the random variable φ_n , an angle relative to the normal vector at the collision point at which a particle leaves after collision n . It is an easy calculation to show that φ_n is drawn from the distribution $\rho_{v_\perp(n)}(\varphi_n) d\varphi_n = \sqrt{\frac{\beta_i}{\pi} \frac{v_\perp(n)}{\cos^2(\varphi_n)}} e^{-\beta_i v_\perp^2(n) \tan^2(\varphi_n)} d\varphi_n$, which depends on β_i associated with $\partial D_i \ni r_n$ and $v_\perp(n)$. We shall also use notation φ'_n for the incoming angle at the point of collision r_{n+1} . Obviously, r_{n+1} , $v_\perp(n+1)$, and φ'_n are completely determined by r_n , $v_\perp(n)$, and φ_n . Note that $v_\perp(n+1) = \frac{\cos(\varphi'_n)}{\cos(\varphi_n)} v_\perp(n)$.

Denote the transition probability kernel of Φ by $\mathcal{P}((r, v_\perp), \cdot)$, i.e. $\mathcal{P}((r, v_\perp), A) = P(\Phi_n \in A | \Phi_{n-1} = (r, v_\perp))$. We will also use the notation \mathcal{P}^* and \mathcal{P}_* for the operators defined on both the set of bounded measurable function and the set of measures of finite mass by

$$(\mathcal{P}^* f)(x, v_\perp) = \int_{\Omega} f(r', v'_\perp) \mathcal{P}((r, v_\perp), d(r', v'_\perp))$$

$$(\mathcal{P}_* \mu)(A) = \int_{\Omega} \mathcal{P}((r, v_\perp), A) \mu(d(r, v_\perp))$$

Note that the transition probabilities are degenerate: $\mathcal{P}((r, v_\perp), \cdot) = \mathcal{P}_* \delta_{(r, v_\perp)}$ is supported on a family of one dimensional curves in the two dimensional phase space Ω .

Our interest lies in investigating the questions of existence, uniqueness, absolute continuity w.r.t. Lebesgue measure m , ergodicity, and mixing properties of the stationary (invariant) measures for the Markov Chain Φ and the associated Markov process $\tilde{\Phi}_\tau$.

3. Results

We show existence of the invariant measures in the non-equilibrium situation simultaneously with the geometric ergodicity for the Markov chain Φ .

The main result is given by the following Theorem:

Theorem 1. *The Markov Chain Φ admits a unique absolutely continuous invariant probability measure μ . Furthermore, there exist a non-negative function V on Ω , as well as constants $C > 0$ and $\tilde{\gamma} \in (0, 1)$ such that*

$$\sup_{A \subset \Omega} |\mathcal{P}^n((r, v_\perp), A) - \mu(A)| \leq C\tilde{\gamma}^n(1 + V(r, v_\perp))$$

Corollary 1. *The invariant measure of the Markov chain Φ is exponentially mixing, i.e. for any Borel $A, B \in \Omega$,*

$$\sup_{A \subset \Omega} \left| \int_B \mathcal{P}^n((r, v_\perp), A) d\mu - \mu(B)\mu(A) \right| \leq C\tilde{\gamma}^n \int_B (1 + V(r, v_\perp)) d\mu \leq De^{-\alpha n},$$

for $\alpha = -\ln(\tilde{\gamma})$ and $D = C \int_\Omega (1 + V(r, v_\perp)) d\mu < \infty$.

Corollary 1 follows from Theorem 1 once $\int_\Omega (1 + V(r, v_\perp)) d\mu < \infty$ is established. This is one of the results of the Theorem 3.2 in [6].

Corollary 2. *If the probability measure ν on Ω satisfies $\int_\Omega (1 + V(r, v_\perp)) d\nu < \infty$, then*

$$\|\mathcal{P}_*^n \nu - \mu\| \leq \tilde{C}\tilde{\gamma}^n \int_\Omega (1 + V(r, v_\perp)) d\nu,$$

where $\|\cdot\|$ is a bounded variation norm.

By constructing a suspension flow over Φ we obtain existence, uniqueness and absolute continuity of the invariant measure for the Markov process $\tilde{\Phi}_\tau$ both for one and for many particle systems.

Corollary 3. *There exists a unique (ergodic) absolutely continuous invariant probability measure for the Markov process $\tilde{\Phi}_\tau$*

Remark 1. In the equilibrium case, i.e. when $\beta_1 = \beta_2 = \dots = \beta_p = \beta$ the invariant measures can be written down explicitly.

Lemma 1. *The measure μ with density*

$$d\mu = \frac{2\beta}{|\partial\Gamma|} v_\perp e^{-\beta v_\perp^2} dv_\perp dr$$

is invariant for the Markov chain Φ .

We will prove Lemma 1 at the end of section 5.

3.1. Idea of Proof of Theorem 1. To prove Theorem 1 we will use the following formulation of the Harris ergodic theorem for Markov chains also referred to as Geometric Ergodicity Theorem.

Theorem 2. [6] *Assume*

Potential Condition .

There exists a function $V : \Omega \rightarrow [0, \infty)$, $K > 0$ and $\gamma \in (0, 1)$ such that

$$\mathcal{P}^*V(r, v_\perp) \leq \gamma V(r, v_\perp) + K$$

for all $(r, v_\perp) \in \Omega$ and

Minorization Condition .

There exists a probability measure ν supported on \mathcal{C} , N and $\eta_N \in (0, 1)$ such that

$$\inf_{(r, v_\perp) \in \mathcal{C}} \mathcal{P}^N((r, v_\perp), \cdot) \geq \eta_N \nu(\cdot),$$

where $\mathcal{C} = \{(r, v_\perp) \in \Omega : V(r, v_\perp) \leq S\}$ for some $S > 2K/(1 - \gamma)$ where K and γ are the constants from the Potential Condition. In addition, to ensure aperiodicity, we require that the same holds for $N+1$ and some $\eta_{N+1} \in (0, 1)$.

Then Φ admits a unique invariant measure μ . Furthermore, there exist $C > 0$ and $\gamma \in (0, 1)$ such that for all $(r, v_\perp) \in \Omega$

$$\|(\mathcal{P}^{*n}f)(r, v_\perp) - \mu(f)(r, v_\perp)\| \leq C\gamma^n \|f - \mu(f)\|,$$

for all f such that $\|f\| < \infty$, where $\|f\| = \sup_{(r, v_\perp)} \frac{|f(r, v_\perp)|}{1+V(r, v_\perp)}$

In particular, for any $A \subset \Omega$, let $f = \begin{cases} 1, & (r, v_\perp) \in A \\ 0, & (r, v_\perp) \notin A \end{cases}$, then

$$\frac{|\mathcal{P}^n((r, v_\perp), A) - \mu(A)|}{V(r, v_\perp) + 1} \leq \sup_{(r, v_\perp)} \frac{|\mathcal{P}^{*n}(f) - \mu(f)|}{V(r, v_\perp) + 1} \leq C\gamma^n$$

so that

$$\sup_{A \subset \Omega} |\mathcal{P}^n((r, v_\perp), A) - \mu(A)| \leq C\gamma^n (1 + V(r, v_\perp)),$$

which is exactly a conclusion of Theorem 1. We chose a weaker formulation of the theorem in order to make the presentation more intuitive.

To prove Theorem 1 we need to show that the Markov Chain Φ satisfies the conditions of the Harris Ergodic Theorem and that the invariant measure we obtain is absolutely continuous with respect to the Lebesgue measure on Ω . We construct the potential V in section 4 and prove the minorization condition on a level set of V and absolute continuity in section 5.

4. Potential V

Proposition 1. *There exists a function $V : \Omega \rightarrow [0, \infty)$ and constants $K > 0$ and $\gamma \in (0, 1)$ such that*

$$\mathcal{P}^*V(r, v_\perp) = \int_{-\pi/2}^{\pi/2} V(r', v'_\perp) \rho_{v_\perp}(\varphi; \beta_i) d\varphi \leq \gamma V(r, v_\perp) + K, \quad \forall (r, v_\perp) \in \Omega, \quad (1)$$

where

$$\rho_{v_\perp}(\varphi; \beta_i) = \sqrt{\frac{\beta_i}{\pi}} \frac{v_\perp}{\cos^2(\varphi)} e^{-\beta_i v_\perp^2 \tan^2(\varphi)}$$

and r' and v' are the position and the normal velocity at the next collision given r , v_\perp , and φ .

The density $\rho_{v_\perp}(\varphi; \beta_i)$ implicitly depends on r through the inverse temperature of the corresponding thermostat. Since we have only a finite number of thermostats, the β_i are bounded above and below and do not play an important role in the asymptotic analysis. To simplify the notation we will write $\rho_{v_\perp}(\varphi)$ instead of $\rho_{v_\perp}(\varphi; \beta_i)$.

4.1. Heuristics. The inequality (1) ensures that the values of V at the random images of (r, v_\perp) are, on average, smaller than $V(r, v_\perp)$ when $V(r, v_\perp)$ is large, and, on average, smaller than a constant when $V(r, v_\perp)$ is small. This implies the dynamics enters the ‘center’ of the phase space, represented by some level set of V , regularly with tight control on the length of excursions from the ‘center’ [6, 10]. Since we expect that particles reach very low or very high velocities with very low probabilities and would like the system to satisfy the minorization condition on the center, a natural physically meaningful candidate for the center would be a set of states with moderate v_\perp velocities, e.g. $\mathcal{C} = \{(r, v_\perp) : v_\perp^{\min} < v_\perp < v_\perp^{\max}\}$ for some $0 < v_\perp^{\min} < v_\perp^{\max} < \infty$. Thus we are looking for a potential V that takes $O(1)$ values in \mathcal{C} and blows up as $v_\perp \rightarrow 0$ and as $v_\perp \rightarrow \infty$. This is essential in order to satisfy the minorization condition.

First let us investigate heuristically the mechanism that ensures that a potential of this kind should, in principle, satisfy the inequality in Prop. 4.1. Assume for simplicity that V depends on v_\perp only and \mathcal{C} is a level set of V , i.e. $\mathcal{C} = \{(r, v_\perp) : v_\perp^{\min} < v_\perp < v_\perp^{\max}\} = \{(r, v_\perp) : V(v_\perp) < A\}$ for some A . We are interested in the change of potential we can expect if we start with low, moderate and high v_\perp and iterate one step forward along some likely random trajectory. The velocity of the particle between collisions with thermostats is $v = v_\perp / \cos(\varphi)$, where φ is the angle drawn from $\rho_{v_\perp}(\varphi) d\varphi$, and the normal velocity upon collision is $v'_\perp = v \cos(\varphi')$, where φ' is the angle of incidence of the particle that originated at (r, φ) .

- If v_\perp is very small, with large probability v_t is the main contribution to the overall velocity v and $v'_\perp \approx \cos(\varphi') v_t$. Again, with large probability, φ' is bounded away from $\pm \frac{\pi}{2}$ so that v'_\perp is of moderate range. Thus with large probability the value of V drops, i.e. $V(v'_\perp) < V(v_\perp)$.

- If v_\perp is $O(1)$, with large probability the v_t contribution is ‘comparable’ to v_\perp , so v is of the same order of magnitude as v_\perp . Again, with large probability φ and φ' are bounded away from $\pm\frac{\pi}{2}$, and thus v'_\perp is of the same order of magnitude as v_\perp and $V(v'_\perp) = O(1)$.
- If v_\perp is very large: with large probability v_t is negligible and $\varphi \approx 0$. Thus $v'_\perp \approx \cos(\varphi')v_\perp < v_\perp$ provided $\cos(\varphi')$ is bounded away from 0 i.e. with large probability $V(v'_\perp) < V(v_\perp)$.

We conclude that, in principle, a potential V taking $O(1)$ values in C and tending to infinity as $v_\perp \rightarrow 0$ and as $v_\perp \rightarrow \infty$ should satisfy the inequality in Prop. 4.1. The escapes to infinity should be fast enough to ensure large enough drops of averaged value, though not too fast to ensure integrability of the left hand side. There are also several potentially dangerous geometric locations: when v_\perp is very small near locations $r \in \partial\Gamma$ such that $\varphi = -\varphi' = \pm\frac{\pi}{2}$ and when v_\perp is very large near $r \in \partial\Gamma$ such that $\varphi = \varphi' = 0$. The heuristic large probability arguments fail for those. In the next subsection we will present a potential V that depends on v_\perp and show that it satisfies the inequality of Proposition for all small, moderate v_\perp , and large v_\perp , even in the aforementioned dangerous geometric locations.

4.2. Construction of the potential V . A reasonable guess for $V(v_\perp)$ is

$$V(v_\perp) = \begin{cases} e^{\epsilon v_\perp^2}, & v_\perp > v_\perp^{\max}; \\ v_\perp^{-a}, & v_\perp < v_\perp^{\min}; \\ A, & v_\perp^{\min} \leq v_\perp \leq v_\perp^{\max}. \end{cases} \quad (2)$$

where $\epsilon < \beta_{\min} = \min\{\beta_1, \dots, \beta_p\}$ and $0 < a < 2$. Let us fix $a = -1$. We will fix ϵ later. We will also assume that $e^{\epsilon(v_\perp^{\max})^2} = (v_\perp^{\min})^{-1} = A$ so that V is continuous.

We are interested in establishing that for some $\gamma > 1$ and $K < \infty$

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(v'_\perp) \rho_{v_\perp}(\varphi) d\varphi \\ & \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\cos(\varphi') v_\perp}{\cos(\varphi)} \right]^{-1} \rho_{v_\perp}(\varphi) d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(\epsilon \left[\frac{\cos(\varphi') v_\perp}{\cos(\varphi)} \right]^2\right) \rho_{v_\perp}(\varphi) d\varphi + A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho_{v_\perp}(\varphi) d\varphi \\ & \leq \gamma V(v_\perp) + K \end{aligned} \quad (3)$$

4.2.1. Controlling small v_\perp . In this subsection we are going to check the inequality for a potential (2) given (r, v_\perp) , where $v_\perp < v_\perp^{\min}$ for some v_\perp^{\min} small enough. When the random map Φ is applied, we first draw φ from the distribution with density $\rho_{v_\perp}(\varphi)$ and then jump to (r', φ', v'_\perp) , where (r', φ') is the image of (r, φ) under the billiard map and $v'_\perp = \frac{\cos(\varphi')}{\cos(\varphi)} v_\perp$. As φ sweeps from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the billiard map image (r', φ') sweeps through several thermostats (we only count those with positive measure of (r', φ') -images). These thermostats can be

divided into two groups: ‘outer’ thermostats, roughly speaking those that intersect two rays originating at $(r, \pm \frac{\pi}{2})$ and ‘inner’ thermostats, all the rest. Of course, some thermostats in this description may repeat: nevertheless we may classify the thermostat differently depending at which angle we are looking at it.

Figure 3 gives an idea of how $\frac{\cos(\varphi')}{\cos(\varphi)}$ behaves as φ sweeps from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ in a typical case. Each curve represents a thermostat. The figure does not specify which thermostat blocks another, so there is an overlay of the values and we will be integrating over all the overlays since we do not make any assumptions on geometry; of course, only one branch works for each value of φ in the real setting.

Fix $0 < \gamma_s < 1$. We would like to split the first integral in (3) into two parts depending whether $\frac{\cos(\varphi')}{\cos(\varphi)} < \frac{1}{\gamma_s}$ or $\frac{\cos(\varphi')}{\cos(\varphi)} \geq \frac{1}{\gamma_s}$.

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} \left[\frac{\cos(\varphi') v_{\perp}}{\cos(\varphi)} \right]^{-1} \sqrt{\frac{\beta_i}{\pi}} \frac{v_{\perp}}{\cos^2(\varphi)} e^{-\beta_i v_{\perp}^2 \tan^2(\varphi)} d\varphi \\
&= v_{\perp}^{-1} \int_{\frac{\cos(\varphi')}{\cos(\varphi)} \geq \frac{1}{\gamma_s}} \left[\frac{\cos(\varphi')}{\cos(\varphi)} \right]^{-1} \sqrt{\frac{\beta_i}{\pi}} \frac{v_{\perp}}{\cos^2(\varphi)} e^{-\beta_i v_{\perp}^2 \tan^2(\varphi)} d\varphi \\
&+ \int_{\frac{\cos(\varphi')}{\cos(\varphi)} < \frac{1}{\gamma_s}} \left[\frac{\cos(\varphi')}{\cos(\varphi)} \right]^{-1} \sqrt{\frac{\beta_i}{\pi}} \frac{1}{\cos^2(\varphi)} e^{-\beta_i v_{\perp}^2 \tan^2(\varphi)} d\varphi \\
&\leq \gamma_s v_{\perp}^{-1} + \sqrt{\frac{\beta_{\max}}{\pi}} \int_{\frac{\cos(\varphi')}{\cos(\varphi)} < \frac{1}{\gamma_s}} \left[\frac{\cos^2(\varphi')}{\cos^2(\varphi)} \right]^{-\frac{1}{2}} \frac{d\varphi}{\cos^2(\varphi)}, \tag{4}
\end{aligned}$$

where $\beta_{\max} = \max\{\beta_1, \dots, \beta_p\}$.

To estimate the second integral in (4), we need to express $\cos(\varphi')$ in terms of φ for each thermostat in range. Given a thermostat D_i let φ_0 be such that corresponding $\varphi'_0 = -\frac{\pi}{2}$ if we assume that all other thermostats are invisible, i.e. the particle can pass through them. See Fig. 4. For some of the ‘outer’ thermostats such φ_0 will not exist: in this case we have to choose φ_0 with corresponding $\varphi'_0 = \frac{\pi}{2}$ and perform a mirror-image parametrization of $\cos(\varphi')$. The computation is essentially the same and thus omitted.

By sine theorem (see Fig 4)

$$\frac{R}{\cos(\varphi - \varphi_0)} = \frac{R - \tau_0 \tan(\varphi - \varphi_0)}{\sin(-\varphi')}$$

Thus

$$\begin{aligned}
\sin(-\varphi') &= \cos(\varphi - \varphi_0) - \frac{\tau_0}{R} \sin(\varphi - \varphi_0) = \cos(\varphi) \cos(\varphi_0) + \sin(\varphi) \sin(\varphi_0) \\
&- \frac{\tau_0}{R} \sin(\varphi) \cos(\varphi_0) + \frac{\tau_0}{R} \cos(\varphi) \sin(\varphi_0) = \frac{1 + w w_0 - \frac{\tau_0}{R} w + \frac{\tau_0}{R} w_0}{\sqrt{w^2 + 1} \sqrt{w_0^2 + 1}}
\end{aligned}$$

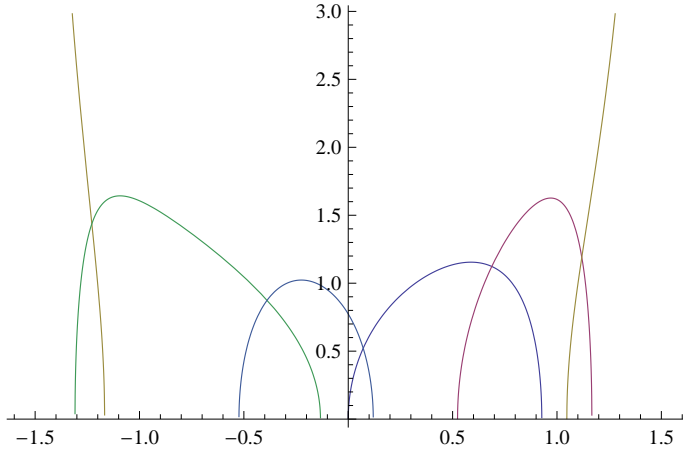


Fig. 3. $\frac{\cos(\varphi')}{\cos(\varphi)}$ for all visible scatterers

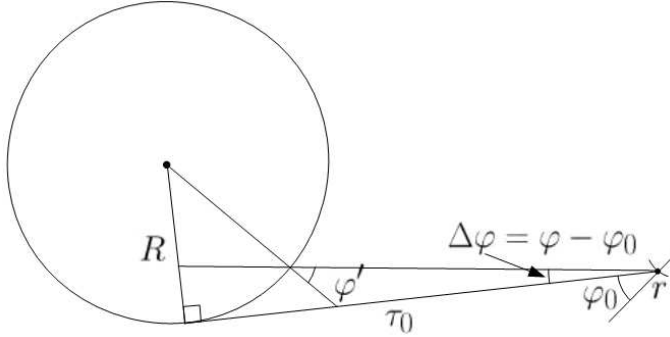


Fig. 4. Parametrization of φ'

here we changed variables $w = \tan(\varphi)$ with $w_0 = \tan(\varphi_0)$.

$$\begin{aligned} \frac{\cos^2(\varphi')}{\cos^2(\varphi)} &= \frac{(w^2 + 1)(w_0^2 + 1) - (1 + ww_0 - \frac{\tau_0}{R}w + \frac{\tau_0}{R}w_0)(1 + ww_0 - \frac{\tau_0}{R}w + \frac{\tau_0}{R}w_0)}{w_0^2 + 1} = \\ &= \frac{1 + 2\frac{\tau_0}{R}w_0 - \frac{\tau_0^2}{R^2}w^2 - 2\frac{w_0 - \frac{\tau_0}{R} - \frac{\tau_0^2}{R^2}w_0 + \frac{\tau_0}{R}w_0^2}{w_0^2 + 1}w + \frac{w_0^2 - 2\frac{\tau_0}{R}w_0 - \frac{\tau_0^2}{R^2}w_0^2}{w_0^2 + 1}}{w_0^2 + 1} \\ &=: aw^2 - 2bw + c \end{aligned}$$

For ‘inner’ thermostats, which we define as thermostats with corresponding $a < 0$, the above expression is a concave down parabola. We are interested in the values between the roots, where the smaller of the roots is w_0 . The maximum value at each parabola is at least 1 since for some φ , the particle hits the thermostat radially, i.e. with $\cos(\varphi') = 1$, so that $\frac{\cos^2(\varphi')}{\cos^2(\varphi)} = \frac{1}{\cos^2(\varphi)} \geq 1$. Let l be the maximal value of the parabola if $aw^2 - 2bw + c = \frac{1}{\gamma_s^2}$ has no real solutions

and $l = \frac{1}{\gamma_s^2}$ otherwise. Denote the smaller or the only root of $aw^2 - 2bw + c = l$ by w_1 . Then

$$\frac{l(w - w_0)}{(w_1 - w_0)} \leq aw^2 - 2bw + c \leq \frac{1}{\gamma_s^2} \quad \text{on } [w_0, w_1] \quad \text{and}$$

$$\int_{w_0}^{w_1} (aw^2 - 2bw + c)^{-1/2} dw \leq \int_{w_0}^{w_1} \left(\frac{l(w - w_0)}{(w_1 - w_0)} \right)^{-1/2} dw = 2(w_1 - w_0)l^{-\frac{1}{2}}$$

It is left to estimate $(w_1 - w_0)$. Simple computation using the values of coefficients a , b , and c ensures that $\sqrt{b^2 - ac} = \frac{\tau_0}{R}$ and that

$$(w_1 - w_0) = \frac{\sqrt{\frac{\tau_0^2}{R^2} + al} - \frac{\tau_0}{R}}{a}$$

has no extremum points for $a \in [-\frac{\tau_0^2}{lR^2}, 0]$. Thus

$$(w_1 - w_0) \leq \max\left\{\frac{Rl}{\tau_0}, \lim_{a \rightarrow 0}(w_1 - w_0)\right\} = \frac{Rl}{\tau_0}$$

We conclude that

$$\int_{w_0}^{w_1} (aw^2 - 2bw + c)^{-1/2} dw \leq 2\frac{Rl}{\tau_0}l^{-\frac{1}{2}} \leq \frac{2R_{\max}}{\tau_{\min}\gamma_s},$$

where $R_{\max} = \max\{R_1, \dots, R_p\}$ and τ_{\min} is the minimum distance of flight.

The situation is clearly symmetric for another set of roots of $aw^2 - 2bw + c = 0$ and $aw^2 - 2bw + c = l$.

For the ‘outer’ thermostats, when $a \geq 0$, $aw^2 - 2bw + c$ is a concave up parabola or a line with slope $2\frac{\tau_0}{R}$ (case $a = 0$) and we are interested in integrating values with $w > w_0$. The derivative of $aw^2 - 2bw + c$ at w_0 is $2\frac{\tau_0}{R}$ and $2\frac{\tau_0}{R}(w - w_0) \leq aw^2 - 2bw + c$ on $[w_0, \infty]$. Let $w_1 = w_0 + \frac{R}{2\tau_0\gamma_s^2}$. Then $aw^2 - 2bw + c > \frac{1}{\gamma_s^2}$ on $(w_1, \infty]$ and

$$\int_{w_0}^{w_1} (aw^2 - 2bw + c)^{-1/2} dw \leq \int_{w_0}^{w_1} \left(2\frac{\tau_0}{R}(w - w_0)\right)^{-1/2} dw = \frac{R}{\tau_0\gamma_s} \leq \frac{R_{\max}}{\tau_{\min}\gamma_s}.$$

Let τ_{\max} be the maximum distance of flight on the bounded horizon table. Then for each point $r \in \partial I$ the number of ‘visible’ thermostats cannot exceed the maximal number S of thermostats in any disk of radius τ_{\max} if we view the positions of thermostats as a periodic configuration on the plane. Therefore $\frac{\cos^2(\varphi)}{\cos^2(\varphi')}$ is less than $\frac{1}{\gamma^2}$ on at most $2S$ intervals and the integral $\int \frac{\cos^2(\varphi')}{\cos^2(\varphi)} dw$ on each is $\leq \frac{2R_{\max}}{\tau_{\min}\gamma_s}$. Therefore

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} \left[\frac{\cos(\varphi') v_{\perp}}{\cos(\varphi)} \right]^{-1} \sqrt{\frac{\beta_i}{\pi}} \frac{v_{\perp}}{\cos^2(\varphi)} e^{-\beta_i v_{\perp}^2 \tan^2(\varphi)} d\varphi \\
& \leq \gamma_s v_{\perp}^{-1} + \sqrt{\frac{\beta_{\max}}{\pi}} \int_{\frac{\cos(\varphi')}{\cos(\varphi)} < \frac{1}{\gamma_s}} \left[\frac{\cos^2(\varphi')}{\cos^2(\varphi)} \right]^{-\frac{1}{2}} \frac{d\varphi}{\cos^2(\varphi)}, \\
& \leq \gamma_s v_{\perp}^{-1} + \frac{2R_{\max}}{\tau_{\min} \gamma_s} \sqrt{\frac{\beta_{\max}}{\pi}} 2S. \tag{5}
\end{aligned}$$

Note that estimate (5) works equally well for all values of v_{\perp} ; we will use it later for large v_{\perp} estimates.

The second integral in the equation (3) can be bounded by using the fact that $(v'_{\perp})^2 \leq v_{\perp}^2 + v_t^2$:

$$\begin{aligned}
& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(\epsilon \left[\frac{\cos(\varphi') v_{\perp}}{\cos(\varphi)} \right]^2) \rho_{v_{\perp}}(\varphi) d\varphi = \int_{-\infty}^{\infty} e^{\epsilon (v'_{\perp})^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t \\
& \leq e^{\epsilon v_{\perp}^2} \int_{-\infty}^{\infty} \sqrt{\frac{\beta_i}{\pi}} \sqrt{\frac{\beta_i - \epsilon}{\beta_i - \epsilon}} e^{-(\beta_i - \epsilon) v_t^2} dv_t = \sqrt{\frac{\beta_i}{\beta_i - \epsilon}} e^{\epsilon v_{\perp}^2} \tag{6}
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(v'_{\perp}) \rho_{v_{\perp}}(\varphi) d\varphi \\
& \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\cos(\varphi') v_{\perp}}{\cos(\varphi)} \right]^{-1} \rho_{v_{\perp}}(\varphi) d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(\epsilon \left[\frac{\cos(\varphi') v_{\perp}}{\cos(\varphi)} \right]^2) \rho_{v_{\perp}}(\varphi) d\varphi + A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho_{v_{\perp}}(\varphi) d\varphi \\
& \leq \gamma_s v_{\perp}^{-1} + \frac{2R_{\max}}{\tau_{\min} \gamma_s} \sqrt{\frac{\beta_{\max}}{\pi}} 2S + \sqrt{\frac{\beta_i}{\beta_i - \epsilon}} \exp[\epsilon (v_{\perp}^{\min})^2] + A = \gamma_s v_{\perp}^{-1} + K_s,
\end{aligned}$$

for any $v_{\perp} < v_{\perp}^{\min}$ for some v_{\perp}^{\min} and $K_s < \infty$.

4.2.2. *Controlling large v_\perp .* To deal with large v_\perp we first note that for any $\varsigma > 0$,

$$\begin{aligned}
& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(v'_\perp) \rho_{v_\perp}(\varphi) d\varphi \\
& \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v'_\perp)^{-1} \rho_{v_\perp}(\varphi) d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\epsilon(v'_\perp)^2} \rho_{v_\perp}(\varphi) d\varphi + A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho_{v_\perp}(\varphi) d\varphi \\
& \leq \gamma_s v_\perp^{-1} + \frac{2R_{\max}}{\tau_{\min} \gamma_s} \sqrt{\frac{\beta_{\max}}{\pi}} 2S + \int_{-\infty}^{\infty} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t + A \\
& \leq \varsigma e^{\epsilon v_\perp} + \int_{-\infty}^{\infty} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t, \tag{7}
\end{aligned}$$

for any $v_\perp > v_\perp^{\max}$, where $v_\perp^{\max} = v_\perp^{\max}(\varsigma)$ is large enough.

Given $\epsilon > 0$ the estimate (6) gives a bound by $\sqrt{\frac{\beta_i}{\beta_i - \epsilon}} e^{\epsilon v_\perp^{\max}}$ when $v_\perp \leq v_\perp^{\max}$. However, this estimate is too rough to produce uniform $\gamma_l < 1$ for large $v_\perp \geq v_\perp^{\max}$, so we need to be more careful here.

For any $\delta > 0$, and Λ large enough, using Chebyshev's inequality, we have

$$\begin{aligned}
& \int_{|v_t| > \Lambda} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t \\
& \leq e^{\epsilon v_\perp^2} \sqrt{\frac{\beta_i}{\beta_i - \epsilon}} \int_{|v_t| > \Lambda} \sqrt{\frac{\beta_i - \epsilon}{\pi}} e^{-(\beta_i - \epsilon) v_t^2} dv_t \leq e^{\epsilon v_\perp^2} \sqrt{\frac{\beta_i}{\beta_i - \epsilon}} \frac{1}{2(\beta_i - \epsilon) \Lambda^2} < \frac{\delta}{2} e^{\epsilon v_\perp^2}.
\end{aligned}$$

Note that Λ can be chosen independent of ϵ provided $\epsilon < \epsilon_0 < \min\{\beta_1, \dots, \beta_p\}$.

In the remaining the integral

$$\int_{|v_t| \leq \Lambda} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t = \int_{|\tan(\varphi)| = |\frac{v_t}{v_\perp}| \leq \frac{\Lambda}{v_\perp}} e^{\epsilon(v'_\perp)^2} \rho_{v_\perp}(\varphi) d\varphi,$$

when $v_\perp \geq v_\perp^{\max}$ with v_\perp^{\max} large enough, we only integrate over values of $\varphi \approx 0$.

So our new perpendicular velocity $v'_\perp = \frac{\cos(\varphi')}{\cos(\varphi)} v_\perp \approx \cos(\varphi'_0) v_\perp$ depends for the most part on the landing angle φ'_0 which corresponds to $\varphi = 0$. For most locations $r \in \partial\Gamma$, $\cos(\varphi'_0)$ is uniformly bounded away from 1.

Before proceeding with further estimates, we are going to identify troublesome geometric locations. For this purpose we are going to lift the periodic configuration of thermostats from a torus to a plane.

Let I be a line segment connecting the centers of two thermostats D_i and D_j such that I does not intersect with any other thermostats. Then if a particle

originates at $r = I \cap \partial D_i$ in the normal direction, i.e. with $\varphi = 0$, it must hit D_j at r' also in the normal direction, i.e. with $\varphi' = 0$. In addition, the normal velocity does not change along such a trajectory, i.e. $v'_\perp = v_\perp$. Suppose now that initial v_\perp is extremely large. Then the tangential velocity v_t acquired before the flight is, on average, tiny compared to v_\perp and the particle lands very close to r' with $\varphi' \approx 0$; i.e. $v'_\perp \approx v_\perp$ and we do not get much of a drop in v_\perp . And it is clear that the higher the initial velocity v_\perp , the smaller is the drop. This indicates that locations like r and r' are ‘troublesome’ and need to be treated with extra care.

Given a thermostat D_i , consider all line segments I_j on the lift to the plane connecting its center to the centers of other thermostats, possibly intersecting other thermostats, and of length at most $2R_{\max} + \tau_{\max}$, where $R_{\max} = \max\{R_1, \dots, R_p\}$ and τ_{\max} is the maximal length of collision-free path. Note that there are only finitely many such segments I_j since there is an upper bound on the length. Let $r_j = I_j \cap \partial D_i$. If a particle originates from $r \in \partial D_i$ in the normal direction denote the point on next collision with a thermostat by r'_0 (tangential collisions count as collisions).

Let Δr_i be small enough such that for any j the intervals $[r_j - \Delta r_i, r_j + \Delta r_i]$ do not intersect and for all $r \in [r_j - \Delta r_i, r_j + \Delta r_i]$, all r'_0 belong to the same thermostat unless I_j touches a thermostat tangentially. In the latter case we require the same for all $r \in [r_j - \Delta r_i, r_j]$ and for all $r \in [r_j, r_j + \Delta r_i]$.

Let $\tilde{\Delta}r < \min_i \{\frac{R_{\min} \Delta r_i}{2}\}$ and denote by \mathcal{R} the union of all r_j ’s over all thermostats D_i . In addition, consider all segments I connecting the centers of thermostats but not intersecting any other thermostats except tangentially. Denote the collection of possible intersections of such I with $\partial \Gamma$ by $\{r_k\}$. Note that $\{r_k\} \subset \mathcal{R}$ and that if $r \in (r_k - \tilde{\Delta}r, r_k + \tilde{\Delta}r)$, for all φ , such that $|\varphi| \leq \frac{r}{R_{\max}}$, r' belongs to the same thermostat as r'_0 (since $(r_k + 2(r - r_k))'_0$ does).

Lemma 2. *Suppose 1-segment path originates from (r, v_\perp) . Then we can choose $\Delta r \leq \tilde{\Delta}r$ small enough such that*

- if $r \notin (r_k - \Delta r, r_k + \Delta r)$ for all k and $|\varphi| \leq \frac{\Delta r}{R_{\max}}$, then $|\varphi'| \geq \frac{\Delta r}{2R_{\max}}$.
- if $r \in (r_k - \Delta r, r_k + \Delta r)$ for some k and $|\varphi| \leq \frac{|r - r_k|}{R_{\max}}$, then $|\varphi'| \geq \frac{|r - r_k|}{2R_{\max}}$.

Proof. Choose $\Delta r \leq \tilde{\Delta}r$ such that $\sin(\frac{\Delta r}{R_{\min}}) \geq \frac{\Delta r}{2R_{\min}}$. We are going to focus on the first statement. The proof of the second statement is essentially the same noting that the situation with tangential collision would only give a better estimate.

Denote the line segment from (r, φ) to (r', φ') by γ , the thermostat containing r by D , the thermostat containing r' by D' , radius of D by R , and radius of D' by R' . Let I be the line segment connecting the center O of the thermostat D to the center O' of the thermostat D' and let $r_I = I \cap \partial D$ and $r'_I = I \cap \partial D'$. Note that $|I| \leq 2R_{\max} + \tau_{\max}$ since $|\gamma| \leq \tau_{\max}$. By assumptions on Δr , we conclude that $|r - r_I| \geq \Delta r$.

Let J be the line segment parallel to I originating at r and ending at some r'_\parallel . Denote the angle J forms with the normal to ∂D by φ_0 and with the normal to the disk of next collision by φ'_0 . Then $|\varphi_0| = \frac{|r - r_I|}{R} \geq \frac{\Delta r}{R} \geq |\varphi|$. Therefore $r'_\parallel \in \partial D'$ and $|\varphi'| \geq |\varphi'_0|$.

Then $|\varphi'| \geq |\varphi'_0| = |\angle r'_I O' r'_\parallel| \geq |\sin(\angle r'_I O' r'_\parallel)| = \left| \frac{R \sin(\angle r_I O r)}{R'} \right| \geq \frac{R \Delta r}{R' 2R} \geq \frac{\Delta r}{2R_{\max}}.$

If $r \notin (r_k - \Delta r, r_k + \Delta r)$, then for $v_t \leq \Lambda$ and $v_\perp \geq v_\perp^{\max} \geq \frac{\Lambda R_{\max}}{\Delta r}$ large enough, $\varphi \leq \tan(\varphi) \leq \frac{\Lambda}{v_\perp} \leq \frac{\Delta r}{R_{\max}}$ and

$$(v'_\perp)^2 = (v_\perp^2 + v_t^2) \cos^2(\varphi') \leq v_\perp^2 - \sin^2\left(\frac{\Delta r}{2R_{\max}}\right) v_\perp^2 + \Lambda^2 \leq v_\perp^2 - \alpha v_\perp^2,$$

where $\alpha = \frac{1}{2} \sin^2\left(\frac{\Delta r}{2R_{\max}}\right).$

Given $\delta > 0$ and Λ satisfying $\sqrt{\frac{\beta_i}{\beta_i - \epsilon}} \frac{1}{2(\beta_i - \epsilon)\Lambda^2} < \frac{\delta}{2}$ we get

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t \\ & \leq \int_{|v_t| > \Lambda} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t + \int_{|v_t| \leq \Lambda} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t \\ & \leq \frac{\delta}{2} e^{\epsilon v_\perp^2} + e^{-\epsilon \alpha v_\perp^2} e^{\epsilon v_\perp^2} \leq \left(\frac{\delta}{2} + 1 - \delta\right) e^{\epsilon v_\perp^2} = \left(1 - \frac{\delta}{2}\right) e^{\epsilon v_\perp^2} \end{aligned}$$

provided that v_\perp is large enough so that $e^{-\epsilon \alpha v_\perp^2} \leq 1 - \delta$.

Therefore we need to focus our attention on studying the dynamics in regions $(r_k - \Delta r, r_k + \Delta r)$. Let $r \in (r_k - \Delta r, r_k + \Delta r)$ and r' be the image of r given the tangential velocity $v_t \leq \Lambda$. First let us note that we can ignore the situation when intervals I connecting centers of thermostats intersect some other thermostats tangentially since it would only give us a better estimate. Then if $v_\perp \geq v_\perp^{\max} \geq \frac{\Lambda R_{\max}}{\Delta r}$ all appropriate images r' lie on the thermostat D'_k .
Let

$$y = \frac{r v_\perp}{R_k} \quad \text{and} \quad y' = \frac{r' v_\perp}{R'_k}$$

Lemma 3. *Given any $\Lambda > 0$ and $A > 0$, there exist $Y = Y(A, \Lambda)$ and $v_\perp^{\max} = v_\perp^{\max}(A, \Lambda)$ such that*

$$(v'_\perp)^2 - v_\perp^2 \leq -A$$

provided that $|y| \geq Y$, $|v_\perp| \geq v_\perp^{\max}$ and $|v_t| \leq \Lambda$.

Proof. Pick $Y = \max\{R_{\max}\Lambda, 8R_{\max}(A + \Lambda^2)\}$. Then if $|y| \geq Y$, $|r - r_k| \geq \frac{Y}{v_\perp}$. And by Lemma 2, since $|\varphi| \leq |\tan(\varphi)| \leq \frac{\Lambda}{v_\perp} \leq \frac{Y}{R_{\max} v_\perp}$,

$$(v'_\perp)^2 - v_\perp^2 \leq -\sin^2\left(\frac{Y}{2R_{\max} v_\perp}\right) v_\perp^2 + \Lambda^2 \leq -\frac{Y^2}{8R_{\max}^2} + \Lambda^2 \leq -A$$

provided that $v_\perp \geq v_\perp^{\max}$ is large enough.

Then if we choose A such that $e^{-\epsilon A} \leq 1 - \delta$, for $|y| \geq Y$ and $|v_\perp| \geq v_\perp^{\max}$,

$$\int_{-\infty}^{\infty} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t \leq \frac{\delta}{2} e^{\epsilon v_\perp^2} + e^{-\epsilon A} e^{\epsilon v_\perp^2} \leq \left(\frac{\delta}{2} + 1 - \delta\right) e^{\epsilon v_\perp^2} = \left(1 - \frac{\delta}{2}\right) e^{\epsilon v_\perp^2}$$

It remains to treat $|y| \leq Y$ case.

Lemma 4. *Suppose $|v_t| \leq \Lambda$ and $|y| \leq Y$. Then*

$$(v'_\perp)^2 = v_\perp^2 + \left[\frac{d^2}{(R'_k)^2} + \frac{2d}{R'_k}\right] v_t^2 + [R_k + R'_k + d]^2 \frac{y^2}{(R'_k)^2} \\ + [2dR'_k + R_k d + d^2 + (R'_k)^2 + R_k R'_k] \frac{2v_t y}{(R'_k)^2} + \mathcal{E}(v_\perp),$$

where $\mathcal{E}(v_\perp) \rightarrow 0$ as $v_\perp \rightarrow \infty$ uniformly in $\{(y, v_t) : |y| \leq Y, |v_t| \leq \Lambda\}$. Here R_k and R'_k are radii of D_k and D'_k respectively and $d = d_{k,k'}$ is the distance between D_k and D'_k .

Proof. From geometry

$$r' = \frac{R_k + d}{R_k} r + \frac{d}{v_\perp} v_t + O(r^2) \quad \text{as } r \rightarrow 0$$

Then, since $|y| \leq Y$

$$y' = \frac{R_k + d}{R'_k} y + \frac{d}{R'_k} v_t + O\left(\frac{1}{v_\perp}\right) \quad \text{as } v_\perp \rightarrow \infty$$

Note that both $O(r^2)$ and $O(\frac{1}{v_\perp})$ are uniform in $\{(y, v_t) : |y| \leq Y, |v_t| \leq \Lambda\}$.

Let us orient the appropriate line segment I connecting centers of D_k and D'_k vertically and let $\theta = \frac{r}{R_k}$ and $\theta' = \frac{r'}{R'_k}$ be the angles r and r' forms with I . Then

$$v_{vert} = v_\perp \cos(\theta) - v_t \sin(\theta)$$

$$v_{hor} = v_\perp \sin(\theta) + v_t \cos(\theta)$$

and

$$\begin{aligned} v'_\perp &= v_{vert} \cos(\theta) - v_{hor} \sin(\theta) \\ &= v_\perp \cos(\theta) \cos(\theta') - v_t \sin(\theta) \cos(\theta') - v_\perp \sin(\theta) \sin(\theta') - v_t \cos(\theta) \sin(\theta') \\ &= v_\perp \left(1 - \frac{1}{2} \frac{y^2}{v_\perp^2} - \frac{1}{2} \frac{(y')^2}{v_\perp^2}\right) - v_t \frac{y}{v_\perp} - v_\perp \left(\frac{y}{v_\perp} \frac{y'}{v_\perp}\right) - v_t \frac{y'}{v_\perp} + O\left(\frac{1}{v_\perp^2}\right) \\ &= v_\perp + \frac{1}{2} \frac{(y^2 + (y')^2 + 2v_t y + 2y y' + 2v_t y')}{v_\perp} + O\left(\frac{1}{v_\perp^2}\right). \end{aligned}$$

So

$$\begin{aligned} (v'_\perp)^2 - v_\perp^2 &= (y^2 + (y')^2 + 2v_t y + 2y y' + 2v_t y') + O\left(\frac{1}{v_\perp^2}\right) \\ &= \left[y + \frac{R_k + d}{R'_k} y + \frac{d}{R'_k} v_t\right]^2 + 2v_t \left[y + \frac{R_k + d}{R'_k} y + \frac{d}{R'_k} v_t\right] + \mathcal{E}(v_\perp) = \left[\frac{d^2}{(R'_k)^2} + \frac{2d}{R'_k}\right] v_t^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{2dv_t}{R'_k}y + \frac{2(R_k + d)dv_t}{(R'_k)^2}y + 2v_ty + 2v_ty \frac{R_k + d}{R'_k} + [R_k + R'_k + d]^2 \frac{y^2}{(R'_k)^2} + \mathcal{E}(v_\perp) \\
& = [\frac{d^2}{(R'_k)^2} + \frac{2d}{R'_k}]v_t^2 + [2dR'_kv_t + 2(R_k + d)dv_t + 2v_t(R'_k)^2 + 2v_t(R_k + d)R'_k] \frac{y}{(R'_k)^2} \\
& + [R_k + R'_k + d]^2 \frac{y^2}{(R'_k)^2} + \mathcal{E}(v_\perp) = [\frac{d^2}{(R'_k)^2} + \frac{2d}{R'_k}]v_t^2 + [R_k + R'_k + d]^2 \frac{y^2}{(R'_k)^2} \\
& + [2dR'_k + R_kd + d^2 + (R'_k)^2 + R_kR'_k] \frac{2v_ty}{(R'_k)^2} + \mathcal{E}(v_\perp),
\end{aligned}$$

where $\mathcal{E}(v_\perp) = O(v_\perp)$ as $v_\perp \rightarrow \infty$ uniformly in $\{(y, v_t) : |y| \leq Y, |v_t| \leq \Lambda\}$.

We are now ready to estimate $e^{-\epsilon v_\perp} \int_{|v_t| \leq \Lambda} e^{\epsilon(v'_\perp)^2} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t$ for $y \leq Y$ and $v_\perp \geq v_\perp^{\max}$.

$$\begin{aligned}
& \sqrt{\frac{\beta_i}{\pi}} \exp(-\beta_i v_t^2 + \epsilon(v'_\perp)^2 - \epsilon v_\perp^2) = \sqrt{\frac{\beta_i}{\pi}} \exp(-[\beta_i + \epsilon \frac{d^2}{(R'_k)^2} + \epsilon \frac{2d}{R'_k}]v_t^2 \\
& - [(R'_k + d)^2 + dR_k + R_kR'_k] \frac{2yv_t}{(R'_k)^2} \epsilon - [R_k + R'_k + d]^2 \frac{y^2}{(R'_k)^2} \epsilon + \mathcal{E}(v_\perp)) \\
& = \sqrt{\frac{\beta_i}{\beta'_i}} \sqrt{\frac{\beta'_i}{\pi}} \exp(-\beta'_i(v_t + \frac{y\tilde{R}\epsilon}{(R'_k)^2\beta'_i})^2) \\
& \times \exp(-\frac{y^2}{(R'_k)^2} [(R_k + R'_k + d)^2 \epsilon + \mathcal{E}(v_\perp) - \frac{\tilde{R}^2}{(R'_k)^2\beta'_i} \epsilon^2]),
\end{aligned}$$

where $\beta'_i = \beta'_i(\epsilon) = \beta_i + \epsilon \frac{d^2}{(R'_k)^2} + \epsilon \frac{2d}{R'_k}$ and $\tilde{R} = (R'_k + d)^2 + dR_k + R_kR'_k$.

Let $\epsilon \leq \epsilon_0$ be small enough and \tilde{v}_\perp^{\max} large enough such that

$$[R_k + R'_k + d]^2 \epsilon + \mathcal{E}(v_\perp) - \frac{\tilde{R}^2}{(R'_k)^2\beta'_i} \epsilon^2 \geq 0$$

for $v_\perp \geq \tilde{v}_\perp^{\max}$ (can do this by Lemma 4). Then

$$\int_{|v_t| \leq \Lambda} \sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2 + \epsilon(v'_\perp)^2 - \epsilon v_\perp^2} \leq \sqrt{\frac{\beta_i}{\beta'_i}} \int_{-\infty}^{\infty} \sqrt{\frac{\beta'_i}{\pi}} e^{-\beta'_i(v_t + \frac{y\tilde{R}\epsilon}{(R'_k)^2\beta'_i})^2} = \sqrt{\frac{\beta_i}{\beta'_i}}$$

Let

$$\gamma = \max_{\{R_k, R'_k, d_{k,k'}, \beta_i\}} \sqrt{\frac{\beta_i}{\beta'_i}} = \max_{\{R_k, R'_k, d_{k,k'}, \beta_i\}} \sqrt{\frac{\beta_i}{\beta_i + \epsilon \frac{d^2}{(R'_k)^2} + \epsilon \frac{2d}{R'_k}}}.$$

Note that $\gamma < 1$. Choose δ and ς so small that $\gamma + \delta + \varsigma < 1$ and $1 - \frac{\delta}{2} + \varsigma < 1$ and then choose Λ so large that $\sqrt{\frac{\beta_{\min}}{\beta_{\min} - \epsilon}} \frac{1}{2(\beta_{\min} - \epsilon)\Lambda^2} < \frac{\delta}{2}$. Let $\gamma_l = \max\{\gamma +$

$\delta + \varsigma, 1 - \frac{\delta}{2} + \varsigma\}$. Choose $v_{\perp}^{\max} \geq \tilde{v}_{\perp}^{\max}$ that satisfies Lemma 3 and all the other assumptions we made. Then for $v_{\perp} \geq v_{\perp}^{\max}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(v'_{\perp}) \rho_{v_{\perp}}(\varphi) d\varphi \leq \gamma_l \exp^{\epsilon v_{\perp}} = \gamma_l V(v_{\perp}).$$

Let $\gamma = \max\{\gamma_s, \gamma_l\}$ and $K = \max\{K_s, \sqrt{\frac{\beta_{\min}}{\beta_{\min} - \epsilon}} e^{\epsilon(v_{\perp}^{\max})^2}\}$. Then

$$\mathcal{P}^* V(v_{\perp}) = \int_{-\pi/2}^{\pi/2} V(v'_{\perp}) \rho_{v_{\perp}}(\varphi) d\varphi \leq \gamma V(v_{\perp}) + K, \quad \forall (r, v_{\perp}) \in \Omega.$$

5. Minorization Condition

For any $v_{\perp}^{\min} < v_{\perp}^{\max}$, let $\mathcal{C} = \{(r, v_{\perp}) : v_{\perp}^{\min} \leq v_{\perp} \leq v_{\perp}^{\max}\}$ and let ν be uniform probability measure on \mathcal{C} .

Proposition 2. *There exist an integer $N > 0$ and $\eta_N > 0$ such that*

$$\inf_{(r, v_{\perp}) \in \mathcal{C}} \mathcal{P}^N((r, v_{\perp}), \cdot) \geq \eta_N \nu(\cdot)$$

In addition, there exists $\eta_{N+1} > 0$ such that the same holds for $(N+1)^{st}$ push forward.

We are going to prove Prop. 2 in three steps: (i) Prop 3, which guarantees existence of a ‘regular’ N -step sample path from any $(r, v_{\perp}) \in \tilde{\mathcal{C}}$ to any $(r', v'_{\perp}) \in \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}} = \{(r, v_{\perp}) : \tilde{v}_{\perp}^{\min} \leq v_{\perp} \leq \tilde{v}_{\perp}^{\max}\}$ for some $\tilde{v}_{\perp}^{\min} < v_{\perp}^{\min}$ and $\tilde{v}_{\perp}^{\max} > v_{\perp}^{\max}$; (ii) Prop. 4, which gives bounds on the densities when pushed forward along regular paths; and (iii) Prop. 5, which guarantees that for any $(r, v_{\perp}) \in \mathcal{C}$ we can ‘acquire’ density in two steps and the resulting location $(r'', v''_{\perp}) \in \tilde{\mathcal{C}}$. So, originating at $(r, v_{\perp}) \in \mathcal{C}$, we acquire density around $(r'', v''_{\perp}) \in \tilde{\mathcal{C}}$ by Prop 5 and then push it forward along the sample path from Prop 3; and Prop. 4 ensures that we get a uniform lower bound on the density at the final point (r', v'_{\perp})

To state Prop. 3, 4, and 5, we need the following definitions.

Definition 1. *A projected (particle) path is a continuous curve $\gamma : [0, 1] \rightarrow \Gamma$, $s \mapsto \gamma(s)$ that consists of a finite sequence of straight segments meeting at $\partial\Gamma$ with $\gamma(0), \gamma(1) \in \partial\Gamma$.*

Remark 2. Note that a projected path is allowed to have *any* ‘reflections’ off the boundaries of the thermostats $\cup_{j=1}^N \partial D_j$. An example of a projected path is shown in Fig 1.

If we start a projected path with certain normal velocity v_\perp , then all subsequent normal velocities upon collisions are completely determined by the path. More precisely, at k^{th} collision

$$v_\perp^k = \frac{\cos(\varphi'_1) \cdots \cos(\varphi'_k)}{\cos(\varphi_1) \cdots \cos(\varphi_k)} v_\perp,$$

where $\varphi_1, \dots, \varphi_k$ and $\varphi'_1, \dots, \varphi'_k$ are the angles of reflections and incidences for the path. Note also that multiplying the initial v_\perp by a certain factor multiplies all subsequent v_\perp^k by the same factor.

Definition 2. Let γ be a projected path that starts at r and ends at r' . A sample path σ given γ and v_\perp is a path in the phase space that starts at (r, v_\perp) with positions updated according to positions of the path γ and normal velocities updated as above.

Note that given γ and v_\perp , σ is unique and to each sample path σ corresponds a unique projected path.

Definition 3. Given $\delta > 0$, $\Delta r > 0$, and $\Delta\varphi > 0$ a projected path γ is $(\delta, \Delta r, \Delta\varphi)$ -regular if

- All the incoming and outgoing angles are bounded away from $\pm\frac{\pi}{2}$ by δ , i.e. all $\varphi, \varphi' \in (-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta)$.
- For each segment of γ from (r_k, φ_k) to $(r_{k+1}, \varphi'_{k+1})$, any segment originating from $\tilde{r}_k \in (r_k - \Delta r, r_k + \Delta r)$ with angle $\tilde{\varphi}_k \in (\varphi_k - \Delta\varphi, \varphi_k + \Delta\varphi)$ has incoming position \tilde{r}_{k+1} on the same thermostat as r_{k+1} and angle $\tilde{\varphi}'_{k+1} \in (-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta)$; in addition, the same holds for each segment in reverse direction, i.e. from $(r_{k+1}, \varphi'_{k+1})$ to (r_k, φ_k) .

Proposition 3. There exist N , δ_0 , Δr_0 , and $\Delta\varphi_0$ such that given any (r, v_\perp) and (r', v'_\perp) with $\tilde{v}_\perp^{\min} \leq v_\perp, v'_\perp \leq \tilde{v}_\perp^{\max}$, i.e. $(r, v_\perp), (r', v'_\perp) \in \tilde{\mathcal{C}}$, there exists a sample path from (r, v_\perp) to (r', v'_\perp) such that the corresponding projected path

1. is $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular.
2. has exactly N collisions.

Here $\tilde{v}_\perp^{\min} = \cos^2(\frac{\pi}{2} - \delta_0) v_\perp^{\min}$ and $\tilde{v}_\perp^{\max} = \frac{v_\perp^{\max}}{\cos^2(\frac{\pi}{2} - \delta_0)}$.

We are going to prove Prop. 3 in three steps. The first two are covered by the following Lemmas.

Lemma 5. There exist K , δ_0 , Δr_0 , and $\Delta\varphi_0$ such that for any choice of r, r' there exists a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r to r' in Γ with $\leq K$ collisions.

Lemma 6. There exist M , δ_0 , Δr_0 , and $\Delta\varphi_0$ such that given any $(r, v_\perp), (r', v'_\perp) \in \tilde{\mathcal{C}}$ there exists a sample path from (r, v_\perp) to (r', v'_\perp) with $\leq M$ collisions and the corresponding projected path is $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular.

The next two propositions deal with densities and are used in the proof of Prop. 2.

Proposition 4. *Given $\Delta r < \Delta r_0$, Δv_\perp small enough, there exists $\tilde{\eta}_N$ such that if μ is the uniform probability measure in the neighborhood $(r - \Delta r, r + \Delta r) \times (v_\perp - \Delta v_\perp, v_\perp + \Delta v_\perp)$ of $(r, v_\perp) \in \tilde{\mathcal{C}}$, then $\mathcal{P}_*^N \mu \geq \tilde{\eta}_N \tilde{\nu}$, where $\tilde{\nu}$ is the uniform probability measure on $\tilde{\mathcal{C}}$.*

Proposition 5. *There exist η , Δr , and Δv_\perp such that for any $(r, v_\perp) \in \mathcal{C}$, $\mathcal{P}_*^2 \delta_{(r, v_\perp)} \geq \eta \nu_{(r, v_\perp)}$, where $\nu_{(r, v_\perp)}$ is the uniform probability measure on $A_{(r, v_\perp)} = (r'' - \Delta r, r'' + \Delta r) \times (v_\perp'' - \Delta v_\perp, v_\perp'' + \Delta v_\perp)$ for some $(r'', v_\perp'') \in \tilde{\mathcal{C}}$.*

Remark 3. In order to satisfy the conditions of Prop. 2 for both N^{th} and $(N+1)^{st}$ push forward of Φ , we first note that Prop. 3 holds similarly if $(r, v_\perp), (r', v_\perp') \in \hat{\mathcal{C}}$, where $\hat{v}_\perp^{\min} = \cos^3(\frac{\pi}{2} - \delta_0) v_\perp^{\min}$ and $\hat{v}_\perp^{\max} = \frac{v_\perp^{\max}}{\cos^3(\frac{\pi}{2} - \delta_0)}$ since v_\perp^{\min} and v_\perp^{\max} are arbitrary in the first place. Therefore to produce an $N + 1$ segment path from $(r, v_\perp) \in \mathcal{C}$ to $(r', v_\perp') \in \mathcal{C}$, one takes a one step regular sample path from $(r, v_\perp) \in \mathcal{C}$ to some $(\hat{r}, \hat{v}_\perp) \in \hat{\mathcal{C}}$ and then follow an N step regular sample path to (r', v_\perp') . Prop. 4 easily applies for $N + 1$ step regular paths. See proof in subsect. 5.4.

5.1. Proof of Lemma 5. Our periodic bounded horizon billiard configuration with the maximal flight distance denoted by τ_{\max} has the following geometric property:

Lemma 7. *There exist δ_1 , Δr_1 , and $\Delta \varphi_1$ such that for each $r \in \cup_i \partial D_i$ there exists a 1-segment projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular path γ that originates at r .*

Lemma 7 is a rather simple consequence of continuity and compactness. We include a proof for completeness purposes at the end of this subsection.

Definition 4. *Given δ_1 , Δr_1 , and $\Delta \varphi_1$ from Lemma 7, for any $r \in \partial D_i$ let U_r be set of points in ∂D_i that can be reached from r by a 2-segment projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular path.*

Each U_r is non-empty since $r \in U_r$ by Lemma 7 and open by definition of a projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular path. Let J_r be the connected component of U_r that contains r . Then $\cup_r J_r$ is a covering of a compact set ∂D_i . Let $\mathcal{J} = \{J_1, \dots, J_{n_i}\}$ be a finite sub-covering such that no element of \mathcal{J} is fully contained in the union of some other elements of \mathcal{J} . The later minimal property guarantees that each point in ∂D_i is covered by at most 2 elements of \mathcal{J} . Let r_1, \dots, r_{n_i} be the set of points that generated the original $J_i := J_{r_j}$ in the subcovering and $\tilde{r}_1, \dots, \tilde{r}_{n_i}$ be the set of midpoints of all the non-empty intersections of elements of \mathcal{J} .

Further re-order J_1, \dots, J_{n_i} and $\tilde{r}_1, \dots, \tilde{r}_{n_i}$ clockwise such that $\tilde{r}_k \in J_k \cap J_{k+1}$. Then there exists a projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular path $\gamma_i : r_1 \rightarrow s_1 \rightarrow \tilde{r}_1 \rightarrow \tilde{s}_1 \rightarrow r_2 \dots \rightarrow r_{n_i} \rightarrow s_{n_i} \rightarrow \tilde{r}_{n_i} \rightarrow \tilde{s}_{n_i} \rightarrow r_1$ around the thermostat with $s_1, \tilde{s}_1, \dots, s_{n_i}, \tilde{s}_{n_i}$ chosen to satisfy the $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regularity property.

Assume now that r, r' are any two points on ∂D_i . Then for some l and m , $r, r_l \in J_l$ and $r', r_m \in J_m$ and there exist projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular paths $r \rightarrow s \rightarrow r_l$ and $r_m \rightarrow s' \rightarrow r'$. Therefore we can construct a projected $(\delta_1, \Delta r_1, \Delta \varphi_1)$ -regular path from r to r' that starts with $r \rightarrow s \rightarrow r_l$, follows γ_i

from r_l to r_m , and ends with $r_m \rightarrow s' \rightarrow r'$. The number of total collisions for such a path is bounded by $K_i = 4n_i + 4$.

For any two thermostats there is a finite chain of length at most $\leq p$ of thermostats joining these two such that any two neighbors in this chain can be connected by some 1-segment projected paths that do not meet thermostats tangentially. Let $\{\tilde{\gamma}_j\}$ be a finite collection of such 1-segment projected paths such that the above connectivity property is satisfied using paths from $\{\tilde{\gamma}_j\}$ only. Let $\delta_2 = \frac{\pi}{2} - \max_{\varphi}\{|\varphi|\}$, where the maximum is taken over all incoming and outgoing angles φ of all $\tilde{\gamma}_j$. And let Δr_2 and $\Delta\varphi_2$ be such that all $\tilde{\gamma}_j$ are $(\delta_2, \Delta r_2, \Delta\varphi_2)$ -regular.

Then if we choose $\delta_0 = \min\{\delta_1, \delta_2\}$, $\Delta r_0 = \min\{\Delta r_1, \Delta r_2\}$, $\Delta\varphi_0 = \min\{\Delta\varphi_1, \Delta\varphi_2\}$, and $K = \sum_{1 \leq i \leq p} \{K_i\} + p$, where p is the number of thermostats in the system. Then there exists a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from any $r \in \cup_i D_i$ to any $r' \in \cup_i D_i$ with the number of collisions bounded by K .

Proof (of Lemma 7). Assume for $\delta_n = \Delta r^{(n)} = \Delta\varphi^{(n)} = \frac{1}{n}$, there exists a sequence r_n such that no 1-segment projected path originating at r_n is $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})$ -regular. Let r be an accumulation point of r_n . Then by continuity there exists a 1-segment projected path originating from r that is $(\frac{1}{m}, \frac{1}{m}, \frac{1}{m})$ -regular for some m . Since $(\frac{1}{m}, \frac{1}{m}, \frac{1}{m})$ -regularity is an open condition, for each point \tilde{r} in a small neighborhood around r , there exists a 1-segment projected $(\frac{1}{m}, \frac{1}{m}, \frac{1}{m})$ -regular path originating from \tilde{r} , a contradiction.

5.2. Proof of Lemma 6. Suppose we start with r and move to r' along the projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from Lemma 5. Given v_{\perp} , let σ_0 be the corresponding sample path that starts at (r, v_{\perp}) and ends at (r', \tilde{v}_{\perp}) for some \tilde{v}_{\perp} . We would like to extend σ_0 by σ_1 such that σ_1 starts at (r', \tilde{v}_{\perp}) and ends at (r', v'_{\perp}) .

First of all, since the path in Lemma 5 is $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular with $\leq K$ segments \tilde{v}_{\perp} has the following bounds:

$$\cos^K\left(\frac{\pi}{2} - \delta_0\right)\tilde{v}_{\perp}^{\min} \leq \cos^K\left(\frac{\pi}{2} - \delta_0\right)v_{\perp} < \tilde{v}_{\perp} < \frac{v_{\perp}}{\cos^K\left(\frac{\pi}{2} - \delta_0\right)} \leq \frac{\tilde{v}_{\perp}^{\max}}{\cos^K\left(\frac{\pi}{2} - \delta_0\right)}$$

For simplicity, let us remove primes and tilde's for a moment and assume that γ is a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r back to r such that the corresponding sample path given v_{\perp} maps (r, v_{\perp}) to (r, v_{\perp}) . The path γ is easily produced by following several $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular segments forward and then the same segments backwards. It is expected that if we perturb γ keeping the return to r , the final value of the normal velocity \hat{v}_{\perp} would deviate from the initial value v_{\perp} . Repeating the perturbed path several times accumulates the deviation from v_{\perp} with ultimate possibility to reach any values for the final normal velocity.

Note that if γ consists of only 2 segments the perturbation would not give \hat{v}_{\perp} different from v_{\perp} since we have a restriction for the perturbed path to return to the initial point. Nevertheless 4-segment paths have enough flexibility to produce the desired effect.

Each 4-segment projected path $\tilde{\gamma}$ near γ is uniquely determined by the initial position r and the four angles $\varphi_1, \varphi_2, \varphi_3$, and φ_4 at which the particle reflects from the boundaries of the thermostats. For a path like that

$$f_r(\tilde{\gamma}) = f_r(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \frac{\hat{v}_\perp}{v_\perp} = \frac{\cos(\varphi'_1) \cos(\varphi'_2) \cos(\varphi'_3) \cos(\varphi'_4)}{\cos(\varphi_1) \cos(\varphi_2) \cos(\varphi_3) \cos(\varphi_4)},$$

where primes correspond to the angles of incidences for the corresponding segments. The constraint we have on f is that the path returns to the original point r . Note that $f_r(\gamma) = 1$.

Lemma 8. *For any $r \in \partial\Gamma$ there exists $\alpha_r < 1$ such that for any $\alpha \in (\alpha_r, 1]$, there exists a 4-segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r to r with a property that for any corresponding sample path going from (r, v_\perp) to (r, \hat{v}_\perp) , we have $\frac{\hat{v}_\perp}{v_\perp} = \alpha$. Similar path exists with $\frac{\hat{v}_\perp}{v_\perp} = \frac{1}{\alpha}$ property.*

Proof. First we are going to construct $\hat{\gamma}$ such that $f_r(\hat{\gamma}) = 1$ and there exists $\tilde{\gamma}$ near $\hat{\gamma}$ such that $f_r(\tilde{\gamma}) \neq 1$.

Let γ_1 be a 1-segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path originating at r and ending at some r' . Assume $r \in \partial D_i$ and $r' \in \partial D_j$. There exists a segment γ_2 that originates at r and hits D_j tangentially at some r'_\parallel possibly intersecting other thermostats. Since γ_1 is $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular, there exists $r'' \in \partial D_i$ such that $\gamma' := r \rightarrow r' \rightarrow r'' \rightarrow r'_\parallel \rightarrow r$ is a 4-segment path having the first two segments $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular and not intersecting any other thermostats. If $\hat{\gamma} := r \rightarrow r' \rightarrow r'' \rightarrow r' \rightarrow r$, then $f_r(\hat{\gamma}) = 1$ and $f_r(\gamma') = 0$, implying that f_r restricted to paths $r \rightarrow r' \rightarrow r'' \rightarrow \tilde{r} \rightarrow r$ is an analytic function of one variable $\tilde{r} \in D_i$ and is not constant on the connected component of $\hat{\gamma}$. Since $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regularity is an open condition, there exists a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path $\tilde{\gamma} := r \rightarrow r' \rightarrow \tilde{r} \rightarrow r' \rightarrow r$ such that $f_r(\tilde{\gamma}) \neq 1$.

It may happen though that f_r conditioned to return to r has a critical point at $\hat{\gamma}$, in which case we are not guaranteed an open interval $[\alpha_r, 1/\alpha_r]$ of $\frac{\hat{v}_\perp}{v_\perp}$ variation. However, since f_r conditioned to return to r is not constant, there exists a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path γ near $\hat{\gamma}$ such that f_r conditioned to return to r does not have a critical point at γ . The conclusion of Lemma 8 follows.

Given $\alpha < 1$, let R_α be the collection of initial positions $r \in \cup_i D_i$ such that for any $\tilde{\alpha} \in (\alpha, 1)$, there exist two 4-segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular paths from r to r with $\frac{\hat{v}_\perp}{v_\perp} = \alpha$ and $\frac{\hat{v}_\perp}{v_\perp} = \frac{1}{\alpha}$ respectively. Note that this time the 4-segment paths are allowed to reflect off any thermostats. Each R_α is open and, by Lemma 8 and continuity, $\{R_\alpha\}_{\alpha \in (0,1)}$ is a covering of the compact set $\cup_j D_j$. Extracting a finite sub-covering guarantees an upper bound $\alpha_r \leq \alpha_{\max} < 1$ for all $r \in \cup_i D_i$.

Therefore for every $r \in \cup_i D_i$ and $\tilde{\alpha} \in (\alpha_{\max}, \frac{1}{\alpha_{\max}})$, there exist two 4-segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular paths along which v_\perp changes by a fraction $\tilde{\alpha}$ and $\frac{1}{\tilde{\alpha}}$ respectively.

Let n be such that $\alpha_{\max} < [\frac{\tilde{v}_\perp^{\min} \cos^K(\frac{\pi}{2} - \delta_0)}{\tilde{v}_\perp^{\max}}]^{\frac{1}{n}}$ and γ_K be the K -segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r to r' guaranteed by Lemma 5. The

corresponding sample path σ_0 given γ_K and v_\perp maps (r, v_\perp) to (r', \tilde{v}_\perp) for some \tilde{v}_\perp . Let

$$\tilde{\alpha} = [\min\{\frac{\tilde{v}_\perp}{v_\perp}, \frac{v_\perp}{\tilde{v}_\perp}\}]^{\frac{1}{n}}$$

Then $\alpha_{\max} \leq \tilde{\alpha} < 1$ since $\cos^K(\frac{\pi}{2} - \delta_0)\tilde{v}_\perp^{\min} < \tilde{v}_\perp < \frac{\tilde{v}_\perp^{\max}}{\cos^K(\frac{\pi}{2} - \delta_0)}$ and $\tilde{v}_\perp^{\min} \leq v'_\perp \leq \tilde{v}_\perp^{\max}$. Let $\gamma_{\tilde{\alpha}}$ be the 4-segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r' to r' such that $\frac{\tilde{v}_\perp}{v_\perp} = \tilde{\alpha}$ if $\tilde{v}_\perp < v_\perp$ or $\frac{\tilde{v}_\perp}{v_\perp} = \frac{1}{\tilde{\alpha}}$ if $\tilde{v}_\perp > v_\perp$. Then if we extend γ_K by $\gamma_{\tilde{\alpha}}$ repeated n times we obtain a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path with the corresponding sample path σ given v_\perp going from (r, v_\perp) to (r', v'_\perp) as desired. And the total number of collisions of such a path is $\leq M = K + 4n$.

5.3. Proof of Proposition 3. Lemmas 5 and 6 guarantee that there is a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from (r, v_\perp) to (r', v'_\perp) with $\leq M$ collisions provided that $\tilde{v}_\perp^{\min} \leq v_\perp, v'_\perp \leq \tilde{v}_\perp^{\max}$. We would like to find an $N \geq M$ such that for any choice of (r, v_\perp) and (r', v'_\perp) we can construct a path that has exactly N collisions.

Lemma 7 provides us with a simple way to add two segments to a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path with return to the same point r' with the same v'_\perp . However, there may be a parity issue if some of the paths from Lemma 6 have odd number of segments and others even.

For each *odd* k , let C_k be the set of all $r \in \cup_i \partial D_i$ such that there exists a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r to back to r that has k collisions. Each C_k is open.

Claim. $\cup_k C_k$ is an open covering of $\cup_i \partial D_i$ provided that $\delta_0, \Delta r_0$, and $\Delta\varphi_0$ are small enough.

Proof. First of all, we would like to choose $\delta_0, \Delta r_0$, and $\Delta\varphi_0$, smaller than before if necessary, so that $\cup_k C_k$ is non-empty. Let $r \in \partial D_i$ and $r' \in \partial D_j$ be the closest two points connected by a projected path, i.e. $|r' - r| = \tau_{\min}$. Then there exists a rectangle with two of the sides parallel to the segment $r \rightarrow r'$ that meets no other thermostats. Extend the sides perpendicular to $r \rightarrow r'$ until they meet some other thermostats D_k and D_l at points r_k and r_l respectively. If r_k is at a corner of the rectangle, perturb $r \in D_i, r' \in D_j$ and/or $r_k \in D_k$ such that $\tilde{r} \rightarrow \tilde{r}' \rightarrow \tilde{r}_k \rightarrow \tilde{r}$ is a 3-segment projected path with all angles of incidence and refraction not equal to $\pm \frac{\pi}{2}$; otherwise this property is guaranteed without the perturbation. Choosing $\delta_0, \Delta r_0$, and $\Delta\varphi_0$ smaller than before if necessary, we ensure that this path is $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular.

Assume now that some $r_1 \notin \cup_j C_j$ and let $r_2 \in C_k$ for some k . Then Lemma 5 provides a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r_1 to r_2 to which we append a path from r_2 to itself with k thermostat collisions, and then the Lemma 5 path in reverse direction, i.e from r_2 to r_1 . This generates a projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path from r_1 to itself with odd number of collisions with thermostats, a contradiction.

Since $\cup_k C_k$ is an open covering of a compact set $\cup_i \partial D_i$, there exists a finite sub-covering $\{C_{k_1}, \dots, C_{k_m}\}$ with $k_1 < k_2 < \dots < k_m$. Therefore for each $r \in \cup_i \partial D_i$ there exists a k_m -segment projected $(\delta_0, \Delta r_0, \Delta\varphi_0)$ -regular path γ_r^m .

Let us choose N to be even. Then if the path from r to r' given by Lemma 5 contains odd number of segments, append it by $\gamma_{r'}^m$ to produce a projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path with $\leq K + k_m$ segments. Then continue with boosting v_\perp to the desired value v'_\perp as in Lemma 6. This produces a sample path σ such that corresponding projected path has even number of segments and is $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular with $\leq N$ segments for some uniformly chosen even N . If such a path happens to make $< N$ thermostat collisions, we just add to it the required number of back and forth 2-segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular paths guaranteed by Lemma 7.

5.4. Pushing Density Forward: proof of Prop. 4. Given $\Delta r < \Delta r_0$ and $\Delta v_\perp < \frac{1}{2} \tilde{v}_\perp^{\min} \cos^N(\frac{\pi}{2} - \delta_0)$, let μ be the uniform probability measure in the neighborhood $(r - \Delta r, r + \Delta r) \times (v_\perp - \Delta v_\perp, v_\perp + \Delta v_\perp)$ of $(r, v_\perp) \in \tilde{C}$. We would like to show that $\mathcal{P}_*^N \mu \geq \tilde{\eta}_N \tilde{\nu}$, where $\tilde{\nu}$ is the uniform probability measure on \tilde{C} .

The projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path provided by Prop. 3 does not leave the set $\tilde{C}^{\max} = \{(r, v_\perp) : \cos^N(\frac{\pi}{2} - \delta_0) \tilde{v}_\perp^{\min} \leq v_\perp \leq \frac{\tilde{v}_\perp^{\max}}{\cos^N(\frac{\pi}{2} - \delta_0)}\}$. In the following Lemma 9 we will give the desired bound for any two points in \tilde{C}^{\max} that can be connected by a 1-segment $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path. Prop. 4 follows by applying Lemma 9 N times.

Lemma 9. *Given $\Delta r < \Delta r_0$, $\Delta v_\perp < \frac{1}{2} \tilde{v}_\perp^{\min} \cos^N(\frac{\pi}{2} - \delta_0)$ and $\eta > 0$, there exist $\eta', \Delta r', \Delta v'_\perp$ such that if $(r, v_\perp), (r', v'_\perp) \in \tilde{C}^{\max}$ can be connected by a 1-segment $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path and μ is the uniform measure in the neighborhood $(r - \Delta r, r + \Delta r) \times (v_\perp - \Delta v_\perp, v_\perp + \Delta v_\perp)$ of $(r, v_\perp) \in \tilde{C}$ having density η with respect to Lebesgue measure m on Ω , then $\mathcal{P}_* \mu \geq \eta' \mu'$, where μ' is the restriction of m to $(r' - \Delta r', r' + \Delta r') \times (v'_\perp - \Delta v'_\perp, v'_\perp + \Delta v'_\perp)$.*

Proof. When we push μ forward by Φ , we first perturb it in the angle direction after which we obtain a product measure in 3-dimensional space with density $\eta \times \rho_{\tilde{v}_\perp}(\tilde{\varphi})$ at $(\tilde{r}, \tilde{\varphi}, \tilde{v}_\perp) \in (r - \Delta r, r + \Delta r) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (v_\perp - \Delta v_\perp, v_\perp + \Delta v_\perp)$. If we only allow φ to vary by at most $\Delta \varphi_0 < \frac{\delta_0}{2}$, then the lower bound on this density is $\eta \times \rho_{\min}$, where ρ_{\min} is the minimum value of $\rho_{\tilde{v}_\perp}(\tilde{\varphi})$ when $\cos^N(\frac{\pi}{2} - \delta_0) \tilde{v}_\perp^{\min} - \Delta v_\perp \leq \tilde{v}_\perp \leq \frac{\tilde{v}_\perp^{\max}}{\cos^N(\frac{\pi}{2} - \delta_0)} + \Delta v_\perp$ and $|\tilde{\varphi}| \leq \frac{\pi}{2} - \delta_0 + \Delta \varphi_0$. Next we push this perturbed 3-dimensional measure forward by enhanced billiard map i.e. the billiard map with v_\perp as an additional variable. The derivative matrix for such a map is

$$\begin{pmatrix} -\frac{\tau \kappa + \cos(\varphi)}{\cos(\varphi')} & -\frac{\tau}{\cos(\varphi')} & 0 \\ \frac{\tau \kappa \kappa' + \kappa \cos(\varphi') + \kappa' \cos(\varphi)}{\cos(\varphi')} & \frac{\tau \kappa' + \cos(\varphi')}{\cos(\varphi')} & 0 \\ -\frac{\sin(\varphi') \frac{\partial \varphi'}{\partial r} v_\perp}{\cos(\varphi)} & -\frac{\sin(\varphi') \frac{\partial \varphi'}{\partial \varphi} \cos(\varphi) + \cos(\varphi') \sin(\varphi)}{\cos^2(\varphi)} v_\perp & \frac{\cos(\varphi')}{\cos(\varphi)} \end{pmatrix} \quad (8)$$

and has determinant -1 (note the cancelation of the usual billiard map derivative $\frac{\cos(\varphi)}{\cos(\varphi')}$ with the perpendicular velocity change $\frac{\cos(\varphi')}{\cos(\varphi)}$). Therefore the lower bound on the density does not change when pushed forward. Now we need to project this pushed forward 3-dimensional measure back to (r, v_\perp) -plane ensuring certain lower bound on the density of the projected measure. The only thing we need to guarantee is

Claim. There exist $\Delta r'$, $\Delta v'_\perp$, $\Delta \varphi'$ that do not depend on (r, v_\perp) and (r', v'_\perp) such that the pre-image (under the enhanced billiard map) of the $\Delta r' \times \Delta \varphi' \times \Delta v'_\perp$ -neighborhood of (r', φ', v'_\perp) maps into $\Delta r \times \Delta \varphi_0 \times \Delta v_\perp$ -neighborhood of (r, φ, v_\perp) .

Then the lower bound on the density of $\mathcal{P}_*\mu$ in $(r' - \Delta r, r' + \Delta r) \times (v'_\perp - \Delta v'_\perp, v'_\perp + \Delta v'_\perp)$ is $\eta' = \eta \times \rho_{\min} \times 2\Delta \varphi'$.

Proof (of Claim). Given $\Delta r' < \Delta r_0$ and $\Delta \varphi' < \Delta \varphi_0$, let A be the set of all 1-segment projected (backward) paths originating at \tilde{r}' at angle $\tilde{\varphi}'$ with $(\tilde{r}', \tilde{\varphi}') \in (r' - \Delta r', r' + \Delta r') \times (\varphi' - \Delta \varphi', \varphi' + \Delta \varphi')$. We will denote the landing point of such a path by $(\tilde{r}, \tilde{\varphi})$. Also let

$$G_1(\Delta r', \Delta \varphi') = \sup_A |\tilde{r} - r|$$

$$G_2(\Delta r', \Delta \varphi') = \sup_A |\tilde{\varphi} - \varphi|$$

$$G_3(\Delta r', \Delta \varphi') = \sup_A \left| \frac{\cos(\tilde{\varphi})}{\cos(\tilde{\varphi}')} - \frac{\cos(\varphi)}{\cos(\varphi')} \right|$$

$$G_4(\Delta r', \Delta \varphi') = \sup_A \frac{\cos(\tilde{\varphi})}{\cos(\tilde{\varphi}')}$$

Note that for $1 \leq i \leq 3$, $G_i(\Delta r', \Delta \varphi') \rightarrow 0$ uniformly as $(\Delta r', \Delta \varphi') \rightarrow 0$ as long as all angles are bounded away from $\frac{\pi}{2}$, which is ensured by the fact the $r' \rightarrow r$ is a projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path. G_4 is bounded by $1 \leq G_4 \leq 1/\cos(\frac{\pi}{2} - \delta_0)$. Therefore we can always choose $\Delta r'$ and $\Delta \varphi'$ to ensure that

$$G_1(\Delta r', \Delta \varphi') < \Delta r$$

$$G_2(\Delta r', \Delta \varphi') < \Delta \varphi$$

$$G_3(\Delta r', \Delta \varphi') \tilde{v}_\perp^{\max} < \frac{\Delta v_\perp}{2}$$

Now

$$\sup_A |\tilde{v}_\perp - v_\perp| = \sup_A \left| \frac{\cos(\tilde{\varphi})}{\cos(\tilde{\varphi}')} \tilde{v}'_\perp - \frac{\cos(\varphi)}{\cos(\varphi')} v'_\perp \right| \leq G_3(\Delta r', \Delta \varphi') v'_\perp + G_4(\Delta r', \Delta \varphi') \Delta v'_\perp$$

and we want it to be less than Δv_\perp . Choosing

$$\Delta v'_\perp < \frac{\Delta v_\perp}{2G_4(\Delta r', \Delta \varphi')}$$

does the job.

Then for any $\tilde{r}' \in (r' - \Delta r', r' + \Delta r')$, $\tilde{\varphi}' \in (\varphi' - \Delta \varphi', \varphi' + \Delta \varphi')$, and $\tilde{v}'_\perp \in (v'_\perp - \Delta v'_\perp, v'_\perp + \Delta v'_\perp)$, we guarantee that the image of $(\tilde{r}', \tilde{\varphi}', \tilde{v}'_\perp)$ under the enhanced billiard map belongs to the $\Delta r \times \Delta v_\perp \times \Delta \varphi$ -neighborhood of (r, φ, v_\perp) .

5.5. Acquiring density: proof of Prop. 5. We would like to establish that there exist η , Δr , and Δv_\perp such that for any $(r, v_\perp) \in \mathcal{C}$, $\mathcal{P}_*^2 \delta_{(r, v_\perp)} \geq \eta m_{A_{(r, v_\perp)}}$, where m is Lebesgue measure on Ω and $A_{(r, v_\perp)} = (r'' - \Delta r, r'' + \Delta r) \times (v_\perp'' - \Delta v_\perp, v_\perp'' + \Delta v_\perp)$ for some $(r'', v_\perp'') \in \tilde{\mathcal{C}}$.

The 1-step push forward measure $\mathcal{P}_* \delta_{(r, v_\perp)}$ is supported on a finite union of curves L in the phase space Ω and is determined by the distribution $\rho_{v_\perp}(\varphi_1) d\varphi_1$ of the angle φ_1 . It may happen that (r, v_\perp) is a pre-image of (r', v_\perp') for several different values of φ_1 . Counting just one we conclude that the density at each (r', v_\perp') is

$$\geq \frac{1}{\sqrt{(\frac{\partial r'}{\partial \varphi_1})^2 + (\frac{\partial v_\perp'}{\partial \varphi_1})^2}} \times \rho_{v_\perp}(\varphi_1),$$

where $\frac{\partial r'}{\partial \varphi_1}$ and $\frac{\partial v_\perp'}{\partial \varphi_1}$ are entries (1, 2) and (3, 2) respectively of the enhanced billiard derivative matrix (8).

The 2-step push forward measure $\mathcal{P}_*^2 \delta_{(r, v_\perp)}$ is, in addition, determined by the distribution $\rho_{v_\perp'}(\varphi_2)$ of φ_2 , where $v_\perp' = \frac{\cos(\varphi_1')}{\cos(\varphi_1)} v_\perp$. The density $d(\mathcal{P}_*^2 \delta_{(r, v_\perp)})$ of $\mathcal{P}_*^2 \delta_{(r, v_\perp)}$ at the endpoint (r'', v_\perp'') of the sample path starting from (r, v_\perp) and determined by φ_1 and φ_2 satisfies the estimate

$$d(\mathcal{P}_*^2 \delta_{(r, v_\perp)}) \geq \rho_{v_\perp}(\varphi_1) \times \rho_{v_\perp'}(\varphi_2) / \left| \det \begin{pmatrix} \frac{\partial r''}{\partial \varphi_1} & \frac{\partial r''}{\partial \varphi_2} \\ \frac{\partial v_\perp''}{\partial \varphi_1} & \frac{\partial v_\perp''}{\partial \varphi_2} \end{pmatrix} \right|,$$

where the derivatives are computed using (8) and a similar derivative matrix for the second flight. In particular, $\frac{\partial r''}{\partial \varphi_1} = \frac{\partial r''}{\partial r'} \frac{\partial r'}{\partial \varphi_1}$ and $\frac{\partial v_\perp''}{\partial \varphi_1} = \frac{\partial v_\perp''}{\partial r'} \frac{\partial r'}{\partial \varphi_1}$.

Lemma 10. *There exists a uniform lower bound η on $d(\mathcal{P}_*^2 \delta_{(r, v_\perp)})$ at any endpoint (r'', v_\perp'') of a 2-segment sample path for which the corresponding projected path is $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular and initial v_\perp is such that $v_\perp^{\min} \leq v_\perp \leq v_\perp^{\max}$. In addition, the same bound holds at the corresponding endpoints if we allow the angles φ_1 and φ_2 vary within $\Delta \varphi_0$.*

Note that at least one such path exists for any r by Lemma 7.

Proof. By observing the entries of (8) we see that the determinant above is proportional to v_\perp , which is bounded above by v_\perp^{\max} . In addition, the terms in denominators are products of $\cos(\varphi_1)$, $\cos(\varphi_1')$, $\cos(\varphi_2)$, and/or $\cos(\varphi_2')$ of total power no greater than 3. Therefore by $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regularity of the path, the determinant is bounded above by some constant $C = C(\delta_0, v_\perp^{\max})$.

In addition, if we let φ_1 and φ_2 vary by at most $\Delta \varphi_0$, we get a similar bounding constant. The product $\rho_{v_\perp}(\varphi_1) \times \rho_{v_\perp'}(\varphi_2)$ is clearly bounded for angle variations within $\Delta \varphi_0$ and $v_\perp^{\min} \leq v_\perp \leq v_\perp^{\max}$. The conclusion follows.

Unfortunately $\Delta \varphi_0$ variation in φ_1 and φ_2 does not always produce nonzero variation in both $\Delta r''$ and $\Delta v_\perp''$. For example, if we start from any location $r \in \partial \Gamma$ and choose $r'' = r$, then we get no variation in v_\perp'' at r'' . Note that $\det \begin{pmatrix} \frac{\partial r''}{\partial \varphi_1} & \frac{\partial r''}{\partial \varphi_2} \\ \frac{\partial v_\perp''}{\partial \varphi_1} & \frac{\partial v_\perp''}{\partial \varphi_2} \end{pmatrix} = 0$ in this case. Indeed, such paths correspond to critical

points of $v''_{\perp}(\varphi_1, \varphi_2)$ constrained by $r''(\varphi_1, \varphi_2) = \text{const}$. In order to guarantee nonzero variation in r'' and v''_{\perp} , for each r , we must find a 2-segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path such that the above determinant is uniformly bounded away from zero.

Lemma 11. *For any r there exists a 2-segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path γ defined by angles of reflection φ_1 and φ_2 with $|J(r, \varphi_1, \varphi_2)| > 0$, where*

$$J(r, \varphi_1, \varphi_2) := \det \begin{pmatrix} \frac{\partial r''}{\partial \varphi_1} & \frac{\partial r''}{\partial \varphi_2} \\ \frac{\partial v''_{\perp}}{\partial \varphi_1} & \frac{\partial v''_{\perp}}{\partial \varphi_2} \end{pmatrix} / v_{\perp}.$$

Proof. Fix the first segment of γ to be any 1-segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path ending at some r' with angle of reflection φ_1 . Then $J(r, \varphi_1, \varphi_2)$ is an analytic function of one variable φ_2 . Similar to the proof of Lemma 8 there exist two 1-segment projected paths γ_1 and γ_2 originating at r' and ending at the same thermostat such that γ_1 is $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular and γ_2 has the angle of incidence $\varphi = \pm \frac{\pi}{2}$. Therefore, $J(r, \varphi_1, \varphi_2)$ is not constant on the connected component and there exists φ_2 such that $J(r, \varphi_1, \varphi_2) > 0$.

For any r , let $f(r) = \sup\{J(r, \varphi_1, \varphi_2)\}$, where \sup is taken along all 2 segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular paths. f is an everywhere positive continuous function on a compact set and therefore there exists the minimum value $f_0 > 0$. For each r , fix a 2-segment projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path $\gamma(r)$ given by $\bar{\varphi}_1(r)$ and $\bar{\varphi}_2(r)$ such that $|J(r, \bar{\varphi}_1(r), \bar{\varphi}_2(r))| > \frac{f_0}{2}$. By the Lagrange multipliers method $r''(\varphi_1, \varphi_2)$ constrained by $v''_{\perp}(\varphi_1, \varphi_2)/v_{\perp} = \text{const}$ does not have a critical point at $(\bar{\varphi}_1(r), \bar{\varphi}_2(r))$ and $v''_{\perp}(\varphi_1, \varphi_2)/v_{\perp}$ constrained by $r''(\varphi_1, \varphi_2) = \text{const}$ does not have a critical point at $(\bar{\varphi}_1(r), \bar{\varphi}_2(r))$.

Let $\epsilon(r) > 0$ be the maximal allowed variation such that any $(\tilde{r}'', \tilde{v}''_{\perp}) \in (r'' - \epsilon, r'' + \epsilon) \times (v''_{\perp} - v_{\perp}\epsilon, v''_{\perp} + v_{\perp}\epsilon)$ is an endpoint of a sample path given by $(r, \tilde{\varphi}_1, \tilde{\varphi}_2, v_{\perp})$ with $\tilde{\varphi}_1 \in (\bar{\varphi}_1(r) - \Delta \varphi_0, \bar{\varphi}_1(r) + \Delta \varphi_0)$, $\tilde{\varphi}_2 \in (\bar{\varphi}_2(r) - \Delta \varphi_0, \bar{\varphi}_2(r) + \Delta \varphi_0)$, and $J(r, \tilde{\varphi}_1(r), \tilde{\varphi}_2(r)) > \frac{f_0}{2}$. $\epsilon(r)$ is a continuous positive function on a compact domain $\partial \Gamma$ and achieves its minimum somewhere. Let $\epsilon = \min_r \epsilon(r)$.

Then for each (r, v_{\perp}) and r'' and v''_{\perp} given by $\gamma(r)$, at each $(\tilde{r}'', \tilde{v}''_{\perp}) \in A_{r, v_{\perp}} = (r'' - \epsilon, r'' + \epsilon) \times (v''_{\perp} - v_{\perp}\epsilon, v''_{\perp} + v_{\perp}\epsilon)$, $d(\mathcal{P}_* \delta_{(r, v_{\perp})}) \geq \eta$.

5.6. Proof of absolute continuity of the invariant measure with respect to Lebesgue measure m on Ω .

Lemma 12. *If μ is invariant under Φ_{τ} and for all $(r, v_{\perp}) \in \Omega$ the absolutely continuous component of $\mathcal{P}_*^2 \delta_{(r, v_{\perp})}$ is nonzero, then $\mu \ll m$.*

The assumption of Lemma 12 follows from the proof of Prop. 5: for each $(r, v_{\perp}) \in \Omega$, we constructed a projected $(\delta_0, \Delta r_0, \Delta \varphi_0)$ -regular path in some neighborhood of which the pushed forward measure is absolutely continuous with respect to m (we do not need a uniform lower bound on the density here).

Proof. Assume μ is not absolutely continuous with respect to m . Then μ can be decomposed as a sum of absolutely continuous and singular components $\mu = \mu_{\ll} + \mu_{\perp}$ with $\mu_{\perp}(\Omega) > 0$. By the invariance of μ

$$\mu_{\ll} + \mu_{\perp} = \mu = \mathcal{P}_*^2 \delta_{(r, v_{\perp})} = (\mathcal{P}_*^2 \mu_{\ll})_{\ll} + (\mathcal{P}_*^2 \mu_{\ll})_{\perp} + (\mathcal{P}_*^2 \mu_{\perp})_{\ll} + (\mathcal{P}_*^2 \mu_{\perp})_{\perp} \quad (9)$$

Any push forward of absolutely continuous measure is absolutely continuous, which follows from the proof of Prop. 4 if we drop the regularity bounds and only keep track of absolute continuity. Therefore $(\mathcal{P}_*^2 \mu_{\ll})_{\perp} = 0$ and $\mu_{\ll} = (\mathcal{P}_*^2 \mu_{\ll})_{\ll}$. In addition, Prop. 5 guarantees that $(\mathcal{P}_*^2 \mu_{\perp})_{\perp} > 0$, since we assumed that $\mu_{\perp}(\Omega) > 0$. Therefore

$$\mu_{\ll}(\Omega) < (\mathcal{P}_*^2 \mu_{\ll})_{\ll}(\Omega) + (\mathcal{P}_*^2 \mu_{\perp})_{\ll}(\Omega),$$

which is a contradiction since absolutely continuous part on the left hand side of (9) must be equal to the absolutely continuous part on the right hand side.

5.7. Proof of Lemma 1. Let us first lift the density $\frac{2\beta}{|\partial T|} v_{\perp} e^{-\beta v_{\perp}^2}$ to the 3-dimensional space by sampling φ from $\rho_{v_{\perp}}(\varphi)$: the resulting density at (r, φ, v_{\perp}) is

$$2\beta v_{\perp} e^{-\beta v_{\perp}^2} \rho_{v_{\perp}}(\varphi).$$

We push the measure with this density forward under the enhanced billiard map, the Jacobian of this map is equal to 1 (see Formula (8)), so the value of the density of the pushed forward measure does not change at the corresponding new location $(r', \varphi', v'_{\perp})$.

To find the value of the density of the pushed forward measure at (r', v'_{\perp}) we just need to integrate over all possible values at (r, φ, v_{\perp}) that map to $(r', \varphi', v'_{\perp})$ as φ' ranges over $(-\frac{\pi}{2}, \frac{\pi}{2})$. Note that $v'_{\perp} = \frac{\cos(\varphi')}{\cos(\varphi)}$. Then we obtain

$$\begin{aligned} d(\mathcal{P}_* \mu) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\beta v_{\perp} e^{-\beta v_{\perp}^2} \sqrt{\frac{\beta}{\pi}} \frac{v_{\perp}}{\cos^2(\varphi)} e^{-\beta v_{\perp}^2 \tan^2(\varphi)} d\varphi' \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\beta \sqrt{\frac{\beta}{\pi}} \frac{(v'_{\perp})^2}{\cos^2(\varphi')} e^{-\beta (v'_{\perp})^2 / \cos^2(\varphi')} d\varphi' = 2\beta v'_{\perp} e^{-\beta (v'_{\perp})^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du \\ &= 2\beta v'_{\perp} e^{-\beta (v'_{\perp})^2} \end{aligned}$$

as desired. In the computation used a change of variables $u = \sqrt{\beta} v'_{\perp} \tan(\varphi')$.

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